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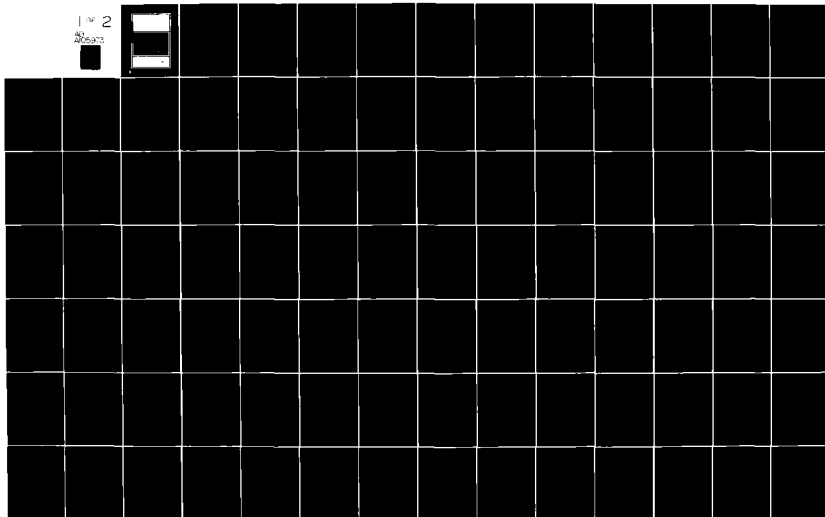
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MULTI-VARIABLE ANALYSIS AND DESIGN TECHNIQUES

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AGARD LECTURE SERIES No. 117

## Multi-Variable Analysis and Design Techniques

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## PREFACE

The Lecture Series is intended to provide the basic theories and concepts involved in the design of advanced guidance and control systems employing state-space and multi-variable design methods. An intrinsic part of this Lecture Series will be computer-aided and graphical techniques that can be employed in preliminary design and related analysis methods. This will provide one document which covers the necessary design background and state-of-the-art involved in the application of advancing technologies.

**Among the main topics to be reviewed are:**

Analysis and Synthesis Techniques  
Application of Observer and Estimation Principles  
Computer-Aided Design and Analysis Methods  
System Simulation Techniques  
Tests Evaluation and Validation

There will be a round-table discussion at the end of the presentations during which comments and suggestions will be invited from participants.

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## The Need for Multivariable Design and Analysis Techniques

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### Summary

Because of escalating performance demands, modern day warfare systems are becoming increasingly complex. Demands for higher accuracy, improved reliability/survivability, and more automation are placing increased emphasis on the control function for successful operation. Nowhere is this increased emphasis more evident than in the control functions required in today's and future aircraft weapon systems.

Modifications to airframe designs directed at increased maneuverability and reduced weight are placing stringent demands on flight control systems. This is particularly true for advanced fighter aircraft which may possess relaxed static stability, additional surfaces for direct force control, and active structural control requirements. Fly-by-wire systems, particularly digital systems, have provided the flexibility to not only accommodate but influence airframe design modifications and have led to the control configured vehicle (CCV) concept. These advances in air vehicle design and flight control system implementation have begun to overwhelm traditional design and analysis techniques which are most effective on aircraft applications with few surfaces, little dynamic coupling, separation between flight control and other avionic subsystem bandwidths and analog system implementations.

In addition, stability and control design specifications have become inadequate in dealing with statically unstable vehicles, multiple control loops and high dynamic coupling. Better control system design and analysis techniques are needed which address the multi-input closely coupled dynamic nature of today's and tomorrow's aircraft weapon systems.

### 1.0 Introduction

A multivariable flight control system is one in which there are multiple interacting control loops. This interaction is dictated by the dynamic coupling resulting from the aircraft and propulsion system design. Until recently, aircraft were designed to minimize dynamic coupling. Propulsion systems were designed based primarily on forward velocity requirements.

For the design of flight control systems with little dynamic coupling, traditional or so called classical design and analysis techniques are adequate and provide key insights into the fundamental design issues of feedback control systems. The design and analysis techniques discussed in subsequent chapters are directed at systems whose control loop interaction extend the utility of classical techniques to the point where they are not only cumbersome to use as design tools but produce flight control system designs with undesirable performance characteristics. The inadequacies of classical design techniques are by no means accepted facts. There has been continuous debate over the last two decades as to the utility of classical techniques versus the utility of non-classical or modern techniques.

It is useful to view the utility issue from the perspective of the flight control system application, particularly as flight control systems have evolved over the years. In depicting that history, the essential items to consider are

- the aircraft application
- the performance requirements
- the control approach

These items determine the utility of the design and analysis techniques.

The first row of Table 1 represents the aircraft application, performance requirements, control approach, and design and analysis techniques for early aircraft up through aircraft built in the 1950's. For these applications the airframe was designed to provide stability and control for the three attitude degrees of freedom and the propulsion system was designed for speed control. The control approach was open loop and design and analysis techniques were airframe oriented. Feedback control design techniques representative of classical techniques are shown for systems described by row 2 of Table 2. In this case also, the airframe was designed to provide 3 DOF attitude stability and control and the propulsion system provide speed control. Inadequate airframe designs or the promise of improved performance resulted in feedback systems which were used to augment stability. The most prevalent example of such a stability augmentation system is a yaw damper. The introduction of feedback control required additional design and analysis techniques, particularly those which addressed

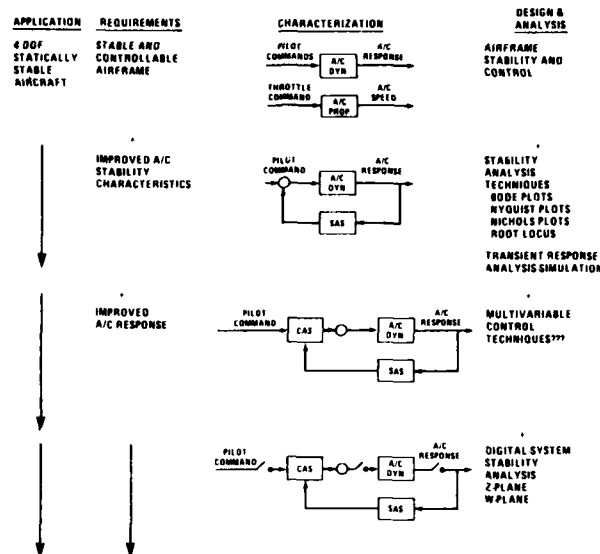


Table 1. Aircraft Control Systems Early To Current Operational Systems

the stability characteristics of feedback control systems. Because the feedback control design was very simple, involving only one sensor variable and one surface command, classical techniques were very effective and led to the acceptance of stability margins as flight control system design specifications.

Additional demands were then placed on flight control systems in the form of command augmentation systems as shown in row 3 of Table 1. The airframe application still remained the same with or without a need for stability augmentation. Handling quality tests determined that command augmentation provided better handling qualities as exemplified by the acceptance of rate command systems or C\* systems. Traditional design techniques were still adequate for design since despite an increase in the number of sensor and surface pairs, the design could be performed one pair at a time because of the loose dynamic coupling.

The introduction of command augmentation, however, initiated the application of modern multivariable techniques, particularly the following approaches. These techniques promised to facilitate flight control system design thus producing better designs. Despite the promise, they were not widely accepted by practical control system designers.

Fly-by-wire systems, particularly digital FBW systems, as shown in row 4 of Table 1, brought new issues to flight control design. New techniques were developed and utilized to insure that digital systems performed as closely as possible to their analog counterparts. In addition the availability of a digital computer and its associated "unlimited" computational capability on board the aircraft encouraged more application of modern techniques which promised better performance. Again these techniques were not widely accepted by practical control designers.

The systems described by row 4 of Table 1 represent the state of the art of today's production aircraft. Table 2 presents characterizations for current experimental and prototype aircraft and projected production systems. The introduction of the control configured vehicle (CCV) concept has had a dramatic effect on flight control systems. In a CCV aircraft, the flight control system is not merely augmenting stability or improving performance, but is providing a flight critical stabilizing and control function. The criticality of the flight control system in a CCV application has intensified the need for efficient and reliable design and analyses techniques. CCV aircraft, in themselves however, do not possess dynamic coupling levels which make classical design techniques intractable. In addition, classical techniques directly address stability issues and have therefore been much more attractive to a designer for CCV control design.

Direct force control, made possible by additional surfaces and thrust vectoring of the propulsion system, as characterized by row 2 of Table 2 introduced a flight control design application that benefits from multivariable control techniques. In this application, it is difficult to eliminate closely coupled dynamics in the airframe design. Interaction with the propulsion system can magnify the coupling. For this application, the large number of control inputs and the close coupling of dynamics can easily overwhelm classical "one loop at a time" techniques.

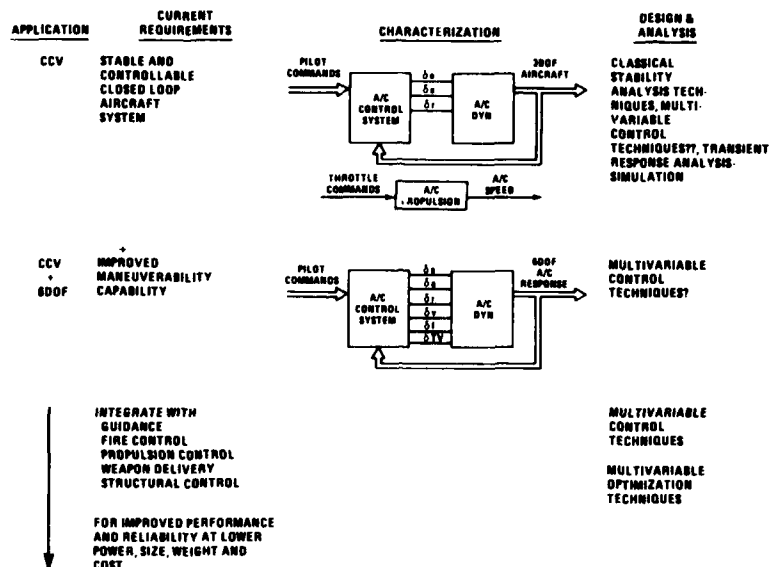


Table 2. Aircraft Control System Early To Current Operational Systems

The problem is projected to worsen with requirements to "integrate" flight control and other avionics subsystems as shown in Row 3 of Table 2 and conceptually in Figure 1. This integration will dramatically increase the number of inputs the control system must handle and introduces the potential for interaction among loops never before encountered in operational systems.

In the sections that follow, we will address the need for multivariable design and analysis techniques. In Section 2.0, a statement of the multivariable control problem will be presented. Section 3.0 will discuss the requirements of multivariable design and analysis techniques for successfully addressing the multivariable control problem. Section 4.0 will present two examples which illustrate the need for better techniques. Summary and conclusions will be presented in Section 5.0.

## 2.0 The Multivariable Control Problem

The multivariable control problem is inherently a feedback control problem. As such, there are two key aspects which will determine the utility of feedback control design and analysis approaches:

- the control application and its representation
- the control design criteria

The elements of each of these aspects will now be discussed.

### 2.1 The control application and its representations

Before beginning any control system design and analysis, it is critical to have a thorough understanding of the nature and requirements of the control application. The understanding of the application can assume different forms and can be obtained from a variety of sources. The basic sources are

- observations of the behavior of the application to be controlled or a similar application
- an analytic understanding of the fundamental behavior of an application deduced from laws of physics, chemistry, etc.

The information from one or both of these sources can be assembled in a form that can be used to predict how the application will behave. These forms can vary in sophistication from "gut feels", to laboratory test beds, to pilot plant or prototypes to complex mathematical representations. All of these assemblages of information or information generators can loosely be termed "models." An essential point to remember, however, is that the model, whatever the form is merely a device to aide in understanding the behavior of the application. The level of understanding and the associated fidelity of the model differ with the nature and the requirements of the application. Over the last twenty-five years, the models used in the aerospace control community have been much more mathematically oriented than in other communities, e.g., process control. The primary reason for this is the lack of a safe, cost effective way to gather information on operational systems that can then be used to predict system behavior. The Space Shuttle is a prime example of the difficulty. Since the Shuttle

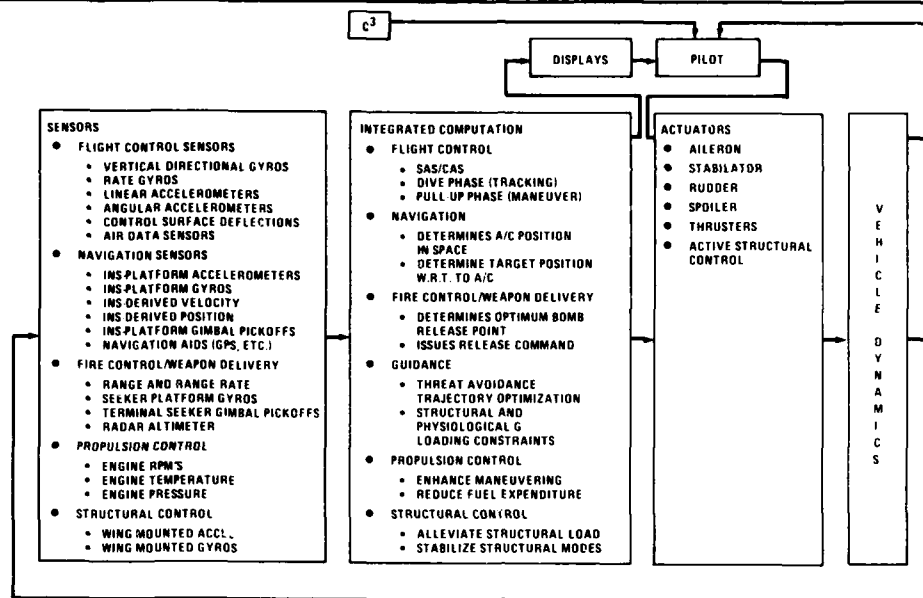


Figure 1. Integrated System Representation

is basically a CCV, the flight control system had to be operational on the very first flight under consequence of loss of vehicle and crew had it failed. With the Shuttle and other CCV's depending on their stability characteristics, open loop flight testing is impossible.

What constitutes a good model? The following rules can be used as guidelines for model development.

1. The fidelity of the model should be tailored to the stage of control system design. For example, in the preliminary design stage, the thrust of the modeling effort should be on major control design drivers leading to a simple but accurate, within limits, model upon which numerous control design tradeoffs can be performed. At the acceptance stage, of course, the model would be of higher fidelity with less tradeoff analysis expected.
2. Modeling approximations should be clearly recognized. Any model is an approximation to an actual system. Every attempt should be made to recognize the impact of the approximations made. The standard mathematical approximations made for flight control system design are
  - linearization of non linear dynamics about selected operating points
  - neglect of some non linearities (e.g., control saturation)
  - neglect of high frequency dynamics.
 These approximations are well understood and in general their impact can be quantified. The number of approximations are reduced as the design proceeds through each development phase.
3. Modeling uncertainties should be recognized. A model uncertainty is different than a modeling approximation. Modeling approximations refer to those actions knowingly taken by a control designer to expedite the control design task. For example, linearizing a non-linear model about an operating point is a modeling approximation. Many times, however, the best mathematical models that can be constructed to represent the behavior of a system still contain a great deal of uncertainty. The uncertainty may arise from the lack of similar systems which could be used for behavioral comparisons or from deficiencies in the understanding of the physical or chemical phenomena involved. As discussed in Section 4.2, the models used for the Space Shuttle Flight Control design had a great deal of model uncertainty. To deal with these uncertainties, the potential variations in uncertain modeling variables should be specified.
4. The consequences of modeling approximations and uncertainties should be recognized. Whether a model of system behavior is approximated or whether there is uncertainty in the model, the consequences of the model limitations as they impact control system design must be recognized. For example, truncating high frequency structural dynamic modes at a particular frequency is acceptable if it is recognized that the control system must also attenuate these high frequency dynamics above that frequency or be insensitive to them.
5. The model should cover the operating range of the application. This requires that the design points be chosen so as to uncover and exercise all of the control system design drivers. The number of design points chosen should be sufficient such that the assemblage of the operating ranges of the control systems at each design point cover the total operating range.

Since control system design and analysis techniques generally begin with an understanding of the control problem, which is represented in model form, the model form and characteristics can become an intimate part of design and analysis techniques.

## 2.2 The Control Design Criteria

Design criteria for feedback control systems can be lumped into three broad categories: Performance, Stability, and Sensitivity. In order to set the stage for defining requirements for multivariable design and analysis techniques, a brief review of these criteria is appropriate.

**Performance.** Performance criteria are generally specified as a requirement for the control system to follow commands or reject disturbances. An example of a command following specification is shown in Figure 2. This type of performance criteria which is used in the Space Shuttle program requires that the vehicle response (in the case shown, roll rate) to a step input (full roll stick command) must fall within a specified envelope. Disturbance rejection criteria can take a number of forms. Disturbances are considered to be externally applied, as in the case of turbulence. A typical criteria for turbulence rejection would be to not exceed a specified vehicle acceleration level during turbulence.

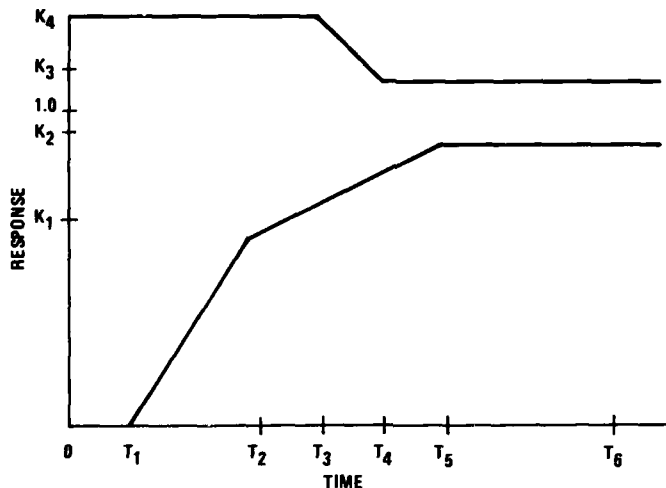


Figure 2. Space Shuttle Entry Step Command Following Response Specifications

**Stability.** The most commonly used stability criteria are stability margins. Stability margins generally refer to the amount of gain or phase variation from a design condition a system employing feedback control can experience before becoming unstable. The gain and phase margin type of stability margin formulation is a consequence of the original techniques (i.e., Bode, Nyquist, Nichols) that were used to analyze feedback control systems. The source of the gain or phase variation, be it modeling approximations, model uncertainty, change in operating conditions or combinations, influences the magnitude of the margins desired. The Space Shuttle stability margin inertia which are fairly typical for flight control systems are shown in Table 3. As will be discussed in subsequent papers, there are limitations associated with using this type of stability criteria.

**Sensitivity.** In early feedback control applications, reducing sensitivity was the only control design criteria, particularly reducing sensitivity to noise. Over the last several years, sensitivity issues have attracted new attention and have generated a new requirement - robustness. A control system is robust (insensitive) if performance is maintained in the presence of variations in the plant from the design model. These variations can be modeling approximations, modeling uncertainties, failures of control elements (sensors, actuators), noise, non-linearities, etc. With the exception of noise, there are no well accepted criteria for dealing with these sensitivity issues at the present time.

These criteria are not mutually exclusive. Good stability characteristics in well designed systems usually are accompanied by good performance. Sensitivity or robustness must be considered relative to the variations from the design model that the control system will experience. The criteria may also be formally or informally applied. For example, satisfying the Shuttle specs shown in Figure 2, was a requirement for control system acceptance. The sensitivity issues, of which there were many in Shuttle, had less rigidly applied criteria associated with them.

	Gain Margin	Phase Margin
Pitch Axis	+ 6 db	30°
Lateral Directional Axis	+ 6 db -12 db	30°

Table 3. Space Shuttle Entry Stability Margin Specifications



### 3.0 Requirements for Multivariable Design and Analysis Techniques

Multivariable design and analysis techniques have no unique requirements over those classical techniques that have been successfully applied to single input-single output (SISO) feedback control system designs. It may be instructive, however, to review the characteristics of SISO techniques that contributed to their utility and widespread acceptance. SISO oriented techniques have the following qualities:

- Provide insight into key feedback control issues
- Easy to use with experience
- Detailed models not required

Techniques with these qualities produced control systems which would satisfy performance specifications with a high degree of confidence and with very little design effort. Probably the most important for these qualities is the amount of insight the technique provides of understanding control system drivers. If the technique directly addresses a design criteria, then it is much easier and quicker to make intelligent design decisions. Techniques that do not provide such insights can be frustrating in their lack of guidance as to what can be done to improve the design. As we stated earlier, control applications have become more complex. Because of this, the same characteristics that provided utility for designing and analyzing SISO systems become less useful for multivariable systems. In fact they become difficult to use and can be misleading in their analysis of key feedback control issues. It is important to understand, however, why they were originally so attractive, because the same qualities should be striven for in multivariable design and analysis techniques.

### 4.0 Illustrative Examples

Two examples illustrate the need for multivariable design and analysis techniques. The first example illustrates the multi-input design and analysis problem associated with an integrated flight and propulsion control system. The second example deals with the inadequacies of current techniques for measuring the stability characteristics of systems with interacting control loop.

#### 4.1 Integrated Flight and Propulsion Control

Air warfare systems are becoming increasingly complex. Figure 3 presents dramatically the increase in control variables that will accompany integration of air warfare functions in future systems. The most significant incremental increase in control variables will occur with the integration of the flight control system and the propulsion control system (engine, inlet, nozzle). This integration promises to improve aircraft maneuvering capability, reduce fuel expenditure, and extend engine lifetimes. The control modes that have been proposed to achieve the advantages of coupling the propulsion and flight control system are

- Energy Management
- 4D Navigation
- Minimum Drag
- Maximum Thrust
- Maneuver Enhancement
- Automatic Envelope Limiting

The fundamental issue in integrating the flight and propulsion control systems is the strong interactions that may be expected between the engines, inlets, and airframe in high performance aircraft, particularly in the supersonic flight regime. Such interaction has been demonstrated in the YF-12 and XB-70 aircraft which experience larger pitch, roll and yaw moments due to airframe/inlet interaction. In the YF-12, for example, the rolling moment produced by the inlet bypass doors is equivalent to that produced by the primary rolling moment generators, the ailerons. The typical maneuvering of higher performance fighters will subject the inlet to large variations in local flow angle, local mach number and pressure all of which will impact engine performance.

In addition to the standard throttle demands placed by the airframe on the engine, fuel consumption will alter the airframe center of gravity requiring retrim. These problems are insignificant, however, when placed in comparison to vectored thrust aircraft like the Harrier and AV-8. Designers forced with integrating complex highly interacting engine inlet and flight control requirements have a formidable task.

In today's turbofan engines, e.g., Pratt and Whitney (P&W) F-100, the fan and compressor margins are defined to accommodate large vehicle maneuvers. In cruise situations improvements in engine efficiency could be realized if the operating points were moved (i.e., uptrimmed). The operating point could then be moved prior to the onset of large maneuvers to maintain margins. Tight control of the engines' operating points and margins requires close cooperation among the various subsystems. This coordination requires the use of multivariable control strategies.

These interactive propulsion and flight control modes listed above will require both inner and outer loop design to fully realize the advantages. Outer loop trajectory commands or optimum set points can be generated using optimization techniques. The multivariable control problem becomes one of shaping command responses to those outer loop commands in a confidently stable control configuration. The type of control and measurements that are available for the problem are shown in Figure 4.

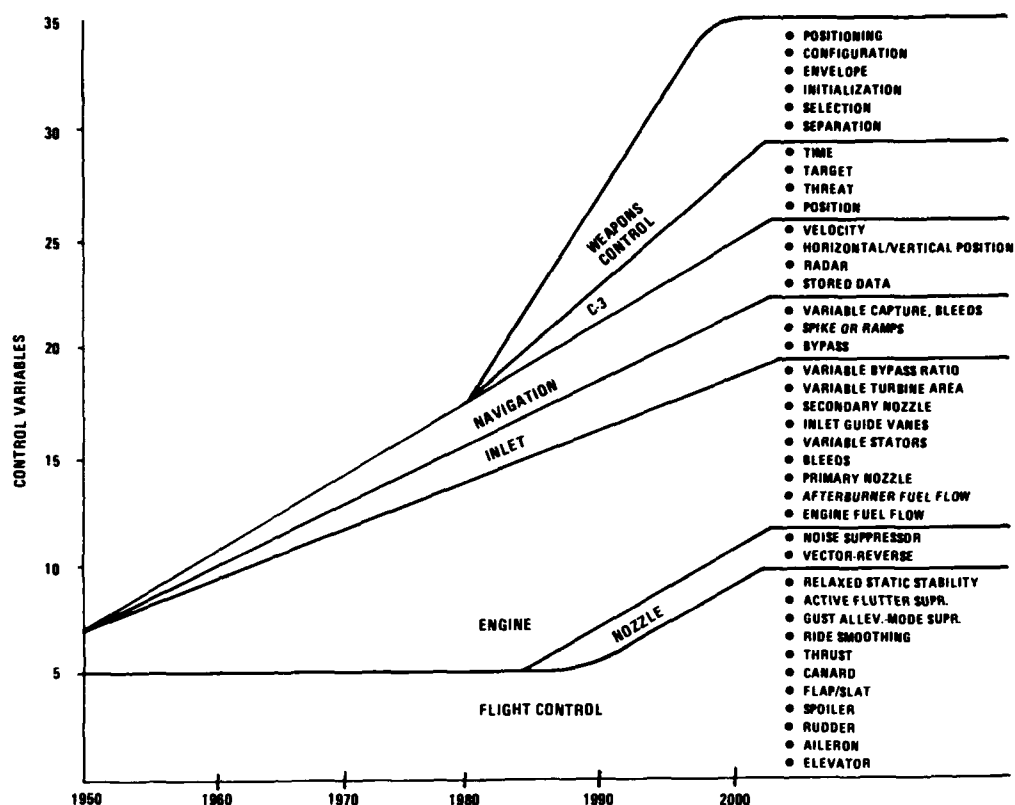


Figure 3. Control Variables are Increasing in Air Warfare Systems

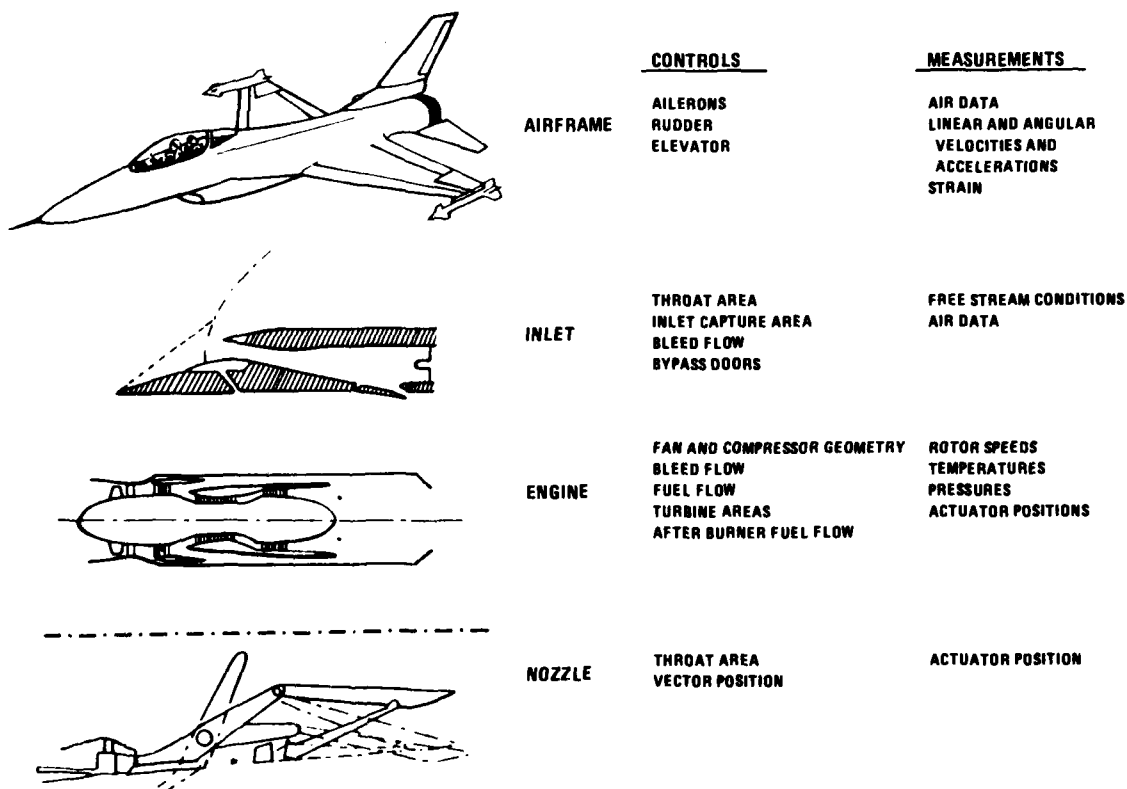


Figure 4. Control and Measurements for Integrated Flight and Propulsion Control

The primary design issues associated with the integrated flight and propulsion system problem reside in the following areas:

**Design Criteria** - Because flight and propulsion systems have traditionally been designed, and operated relatively independently, no design criteria exist for how they should be integrated.

**Design Models** - Again because of the autonomy of the systems, interactive effects are generally only crudely approximated. In addition, the engine control problem is extremely non linear. The flight conditions, of interest, e.g., high angle of attack maneuvering, further complicate the non linear problem.

**Failure Sensitivity** - Because flight and propulsion are more integrated, a failure in a sensor or a controller in either system now has a more significant impact on the total system. Design and verification of integrated flight and propulsion control systems that tolerate failures is a major challenge.

Past Air Force and NASA programs such as the F-111 Integrated Propulsion Control System (IPCS), the YF-12 Cooperative Control (CAPCS) and Various Flight Propulsion and Control Coupling studies have examined some of the issues. The NASA integrated Research Aircraft Control Technology (INTERACT) currently is taking a new look in an attempt to exploit some of the new results in multivariable control.

#### 4.2 The Space Shuttle Flight Control System

Control of the Space Shuttle Orbiter during the entry flight phase is the most demanding flight control problem control designers have ever faced. The major contributors to the problem are a statically unstable vehicle, a vast operating envelope, a large degree of uncertainty in modeling the vehicle behavior, and a control system comprised of a combination of linear (aerodynamic surfaces) and non-linear (reaction control jets) devices.

The large operating envelope is shown in Figure 5. The orbiter control system must operate over velocities from 26,000 FPS to 0 FPS, dynamic pressures of 0 PSF to 400 PSF, and angles of attack from 0° to 50°, all of which directly determine vehicle dynamic response. At the low dynamic pressure portion of the flight, only reaction control jets are employed. However, during the majority of the entry, reaction jets are used in combination with the aerodynamics surfaces for lateral directional control.

The problem that most plagued shuttle control design was uncertainty in modeling the vehicles projected behavior coupled with potential large deviations from a nominal trajectory. The modeling uncertainties were related to aerodynamics, structural flexibility, and flow interaction with reaction jets. The deviations from nominal trajectory were produced by atmospheric conditions (e.g., winds) and potential variations in vehicle drag. These deviations resulted in large variations in dynamic pressure which is a key flight control design element.

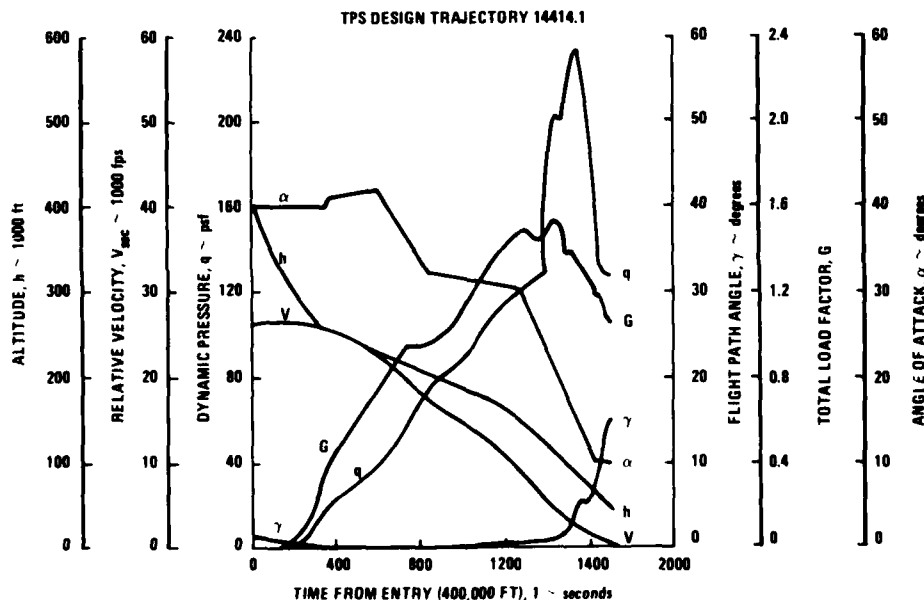


Figure 5. Orbit Entry Parameters

The approach taken to determine the impact of uncertainty involved a number of steps. First, statistical bounds were placed on all uncertain parameters. Statistical bounds were also placed on potential deviations from the nominal trajectory. Second, design criteria were established which required that the control system operate over the statistically defined flight envelope which included not only deviations from a nominal trajectory but variations in vehicles aerodynamics and reaction jet characteristics. Third, the impact of these statistical uncertainties was then analyzed with respect to the command response and stability margin criteria presented in Section 2.0. The analysis approaches employed were

- linear transient response and stability analysis
- primarily linear with selected non linearities transient response analysis
- 6 DOF non linear transient response analysis at fixed flight conditions
- 6 DOF non linear transient response analysis over entry trajectories
- full up simulations
  - automatic
  - man in the loop

As demonstrated by the successful first flight, this exhaustive approach did produce a successful control design.

Because of the uncertainty issue, the Shuttle is a good example to illustrate the inadequacies that currently exist in analyzing the stability characteristics of systems with a large degree of uncertainty. The first inadequacy was the lack of analysis tools that gave the same idea of a "stability margin" for analyzing the effect of the statistically represented uncertainties described above. Because these statistical uncertainties appeared in only parts of the model and in different combinations, standard gain and phase margin analysis techniques did not apply. Analysis required exhaustive, time consuming, costly simulations to investigate all the possible combinations.

A technique which would map an envelope of uncertainty to a stability margin representation would have been extremely useful on the Shuttle control system design. An example is shown in Figure 6 where a two parameter (e.g.,  $C_{N8}$  vs  $C_{L8}$ ) envelope of uncertainty is mapped to a representation of closed loop eigenvalues, for both open and closed loop behavior. A stability margin type measure might be the curve  $P_1'$   $P_2'$  which represents the stability boundaries on the allowable parameter variations. This of course, only represents a two parameter case. For Shuttle control system analysis, the effect of ten to fifteen parameters and all the possible combinations had to be determined.

Another area of concern is the inadequacy of current stability analysis techniques to address the direction of an uncertainty. Classical techniques basically only address magnitude variations in one direction. Preliminary analysis of the first flight Shuttle firing data indicates that the vehicle roll response at low dynamic pressure to a yaw jet firing was adverse rather than proverse as was expected. If this effect is taken into account, the stability margins deteriorate from satisfying specs to a -1 dB low frequency margin.

These two examples indicate a need for new techniques for designing and analyzing control systems with a large number of interacting control loops and levels of uncertainty in predicting system behavior.

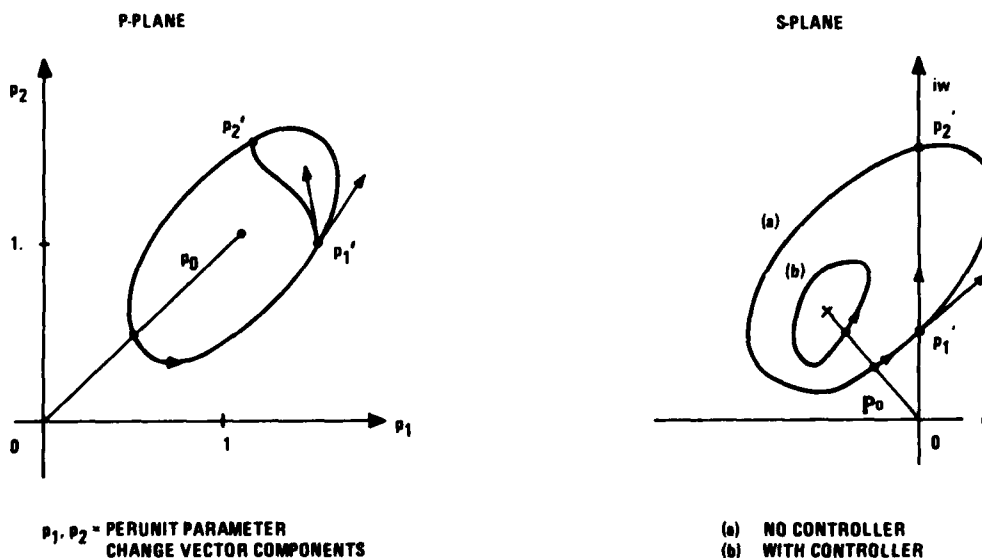


Figure 6. Uncertainty Mapping for Stability Analysis

## 5.0 Summary and Conclusions

A case for the need for new multivariable design and analysis techniques has been presented. Modern air warfare systems demand these techniques in order to satisfy increasingly stringent and complex performance requirements. As we progress into the space environment, the control of large flexible space structures will pose even more severe design challenges particularly in the area of model testing and uncertainty. Our current established techniques are becoming over-taxed and are forcing us to rely more heavily on costly and time consuming simulation for design. Computer aided design approaches are required to assimilate and display the information required for control system design that are easy to use and provide insight into key control design issues.

As exemplified by the existence of the lecture series, there has been considerable interest over the last twenty years in developing new design techniques that address these needs. In the papers that follow, a broad spectrum of techniques will be described that differ widely in design philosophy and approach. It is our hope that the descriptions will provide the essential information for understanding the advantages of one technique over another for particular design problems.

As is the case in the use of any tool, there is generally no one tool which will solve all problems with optimal efficiency. The most efficient tool for a particular application is the one that has been tailored specifically for that application. At the conclusion of this lecture series, we hope to have provided to you a tool box from which you can select the appropriate tool for your control design application.

CHARACTERISTIC AND PRINCIPAL GAINS  
AND PHASES AND THEIR USE AS  
MULTIVARIABLE CONTROL DESIGN TOOLS

by

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SUMMARY

The key to the generalisation of classical frequency-response design techniques to the multivariable case lies in the development of ways of suitably extending the concepts of gain and phase. It is shown how algebraic function theory can be used to generalise Nyquist diagram and Root-Locus diagram techniques for use with systems having many inputs and outputs. This is done in such a way that the main structural features of such diagrams can be related to state-space model parameters. The shortcomings of characteristic gains and phases (used in a generalised Nyquist approach) are then noted. An alternative way of introducing amplitude and phase information, via the polar decomposition of an operator, leads to the introduction of principal gains and principal phases. Their properties are examined and, in particular, it is shown how they may be used to characterise robustness.

Finally a discussion, with an example, is given of the use of these techniques for design purposes.

1. INTRODUCTION

In developing a design technique, one seeks to give practising and experienced engineers a set of manipulative and interpretative tools which will enable them to build up, modify and assess a design put together on the basis of physical reasoning within the guidelines laid down by their engineering experience. Thus, in the development of design techniques, consideration of the way in which a designer interacts with a computer is vitally important. It is imperative to share the burden of work between computer and designer in such a way that each does what they can do best. The development of techniques which lead to a fruitful and effective symbiotic relationship between computer and designer is of great importance in the future development of control engineering practice since it is the only conceivable way in which :

- (i) the designer can deploy his intuitive skill and experience while still making an effective use of advanced theoretical tools; and
- (ii) the immense and ever-increasing computational power of computers can be harnessed in a design context.

A developer of new engineering design techniques has two harsh, and quite different, criteria to satisfy :

- (i) he must provide tools which are acceptable to practising engineers and which are demonstrably useful and effective;
- (ii) he must show that his techniques are based on rigorous and indisputably correct mathematical foundations.

To be effective for an engineering designer, a design technique must make the maximum use of his intuition and experience. The need to provide him with an intuitively appealing medium for communication and visualisation, together with recent developments in computer terminals, places a high premium on techniques using graphics and those which are based on geometrical concepts. What we are concerned with here is the development of an appropriate conceptual framework within which a designer can work. This is based on fairly straightforward concepts in algebraic geometry (the theory of curves) and linear geometry. It is shown how the generalisation of the classical frequency-response tools of Nyquist and Root-Locus diagrams are algebraic curves derived from the spectral analysis of appropriate matrix-valued functions of a complex variable. Many of their key properties can be comprehensively analysed in terms of a state-space model of the vector feedback loop which is being studied. This analysis involves simple linear geometry and is essentially concerned with the way in which certain subspaces, defined via the various operators involved, sit with respect to one another, and with ways of assigning complex frequencies to those subspaces. Finally, the techniques outlined are illustrated by an example of their use in design.

## 2. GENERALISED NYQUIST AND ROOT LOCUS DIAGRAMS: CHARACTERISTIC GAINS AND CHARACTERISTIC PHASES.

The original Nyquist stability criterion owed its central importance in classical feedback control work to the fact that it tested the stability of a proposed feedback loop in terms of a directly measured transmission characteristic [1]. This radical departure of inferring stability from measured signal characteristics instead of from computed dynamical-model characteristics placed complex-function methods in a centrally-important position in the development of control analysis and design techniques up to the late 1940s. The root-locus technique developed in the 1950s [1] added further weight to the arguments for using complex-variable approaches in the analysis and design of feedback systems. At the time of the introduction of the root-locus method, the close link with the Nyquist technique was not appreciated; one of the virtues of the algebraic function approach used here to generalise these techniques to the multivariable case [2] is that it shows how both these classical techniques are essentially different ways of representing curves derived from a single algebraic function relating complex frequency and complex gain.

The dual role of complex numbers is familiar from elementary complex number theory: they can be used to represent both objects or quantities (e.g. vectors in a plane) and operations on objects (e.g. rotation and stretching of vectors in a plane). This dual role is important in classical (i.e. single-input single-output) frequency-response methods where complex numbers are used to represent frequencies (complex numbers) and gains (operations on complex vectors). The two classical approaches to the complex-variable-based analysis of feedback systems:

- (i) study (open-loop) gain as a function of frequency (the Nyquist-Bode approach); or
- (ii) study (closed-loop) frequency as a function of gain (the Evans Root-Locus approach).

Our first objective is to show that there is an intimate relationship between the generalisation of these two approaches to the multivariable (many-input many-output) case and the standard state-space model

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (2.1)$$

In much of what follows the matrix  $D$  will be taken to be zero for simplicity of exposition. However its inclusion at this point gives completeness and symmetry to the development of the general case. [ $A$  is  $n \times n$ ;  $B$  is  $n \times l$ ;  $C$  is  $m \times n$ ;  $D$  is  $m \times l$ . When vector feedback loops are being discussed  $l = m$ ]. Suppose we sever  $m$  feedback connections to an arbitrary linear dynamical system as shown in Fig.1, and that the transmittance between the injection point  $\alpha$  and the return  $\alpha'$  is given by a state-space model of the form shown in equation (2.1).

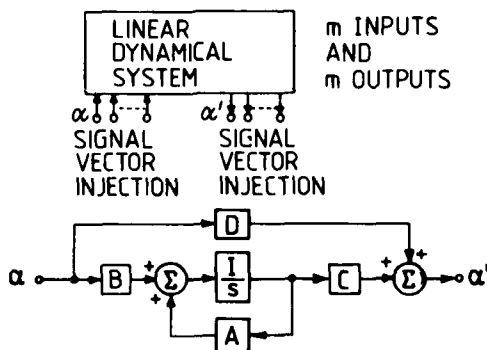


Figure 1

Then, on remaking the feedback connections, we see that the closed-loop characteristic frequencies of this system will be given by the spectrum (set of eigenvalues) of the matrix

$$S = A + B(I_m - D)^{-1}C \quad (2.2)$$

and that the open-loop transfer function matrix relating vector exponential signals injected at  $\alpha$  to the vector exponential signals returned at  $\alpha'$  is

$$G(s) = D + C(sI_n - A)^{-1}B \quad (2.3)$$

( $I_m$  and  $I_n$  are unit matrices of orders  $m$  and  $n$  respectively.) If one compares the right-hand sides of (2.2) and (2.3) they are seen to have a suggestive structural similarity. This becomes an exact structural equivalence if we introduce a variable  $g$  and change (2.2) to the form

$$S(g) = A + B(gI_m - D)^{-1}C \quad (2.4)$$

where the role of the variable  $g$  will emerge shortly. Our aim is to put, in a state-space-model context, the roles of open-loop gain (the key Nyquist-Bode concept) and closed-loop frequency (the key Evans Root-Locus concept) on an exactly equal footing. With this in view we look for a system giving the closed-loop frequency matrix of equation (2.4) and find that it is as shown in Fig.2, where  $g$  is now interpreted as a gain parameter. On redrawing Fig.2 in the form shown in Fig.3 we see that the variables  $s$  and  $g$  have indeed been placed in a strikingly symmetrical relationship.

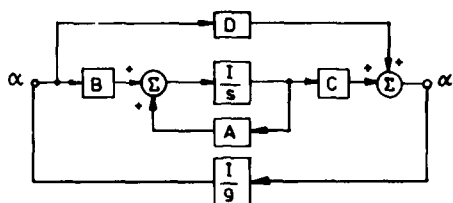


Figure 2

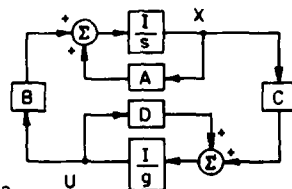


Figure 3

In Fig.3  $s$  is to be interpreted as a complex frequency variable and  $g$  as a complex gain variable. For the related vector feedback loop (with  $m$  inputs and  $m$  outputs) the open-loop gain matrix

$$G(s) = D + C(sI_n - A)^{-1}B$$

describes open-loop gain as a function of imposed frequency  $s$ ; and the closed-loop frequency matrix

$$S(g) = A + B(gI_m - D)^{-1}C$$

describes closed-loop frequency as a function of the imposed gain parameter  $g$ .

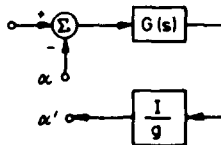


Figure 4



For the negative feedback arrangement of Fig.4 if

$$F(s) = I_m + g^{-1}G(s) \quad (2.5)$$

is the return-difference matrix for the set of feedback loops broken at  $\alpha-\alpha'$  then one can show that [3]

$$\begin{aligned} \frac{\det[sI_n - S(g)] \det[gI_m - G(s)] \det F(s)}{\det[sI_n - S(\infty)] \det[gI_m - G(\infty)] \det F(\infty)} \\ = \frac{\text{CLCP}(s)}{\text{OLCP}(s)} \end{aligned} \quad (2.6)$$

where CLCP(s) and OLCP(s) are the system's closed-loop and open-loop characteristic polynomials respectively. The importance of equation (2.6) is that it tells us that in this archetypal negative feedback situation we may equally well study the effect of closing feedback loops on stability in terms of an open-loop gain description or a closed-loop frequency description. Specifically it shows that, for values of  $s$  not in the spectrum of  $A$  and values of  $g$  not in the spectrum of  $D$

$$\det[sI_n - S(g)] = 0 \iff \det[gI_m - G(s)] = 0 \quad (2.7)$$

This in effect says that a knowledge of how the characteristic values of  $G(s)$  vary as a function of the frequency parameter  $s$  is equivalent (for the purposes of making inferences about closed-loop stability) to a knowledge of how the characteristic values of  $S(g)$  vary as a function of the gain parameter  $g$ .

## 2.1 Characteristic frequencies and characteristic gains.

For a given value of the gain parameter  $g$  the eigenvalues  $\{s_i : i = 1, 2, \dots, n\}$  of  $S(g)$  are the corresponding set of closed-loop characteristic frequencies, obtained from

$$\det[sI_n - S(g)] = 0 \quad (2.8)$$

For a given value of the frequency parameter  $s$  the eigenvalues  $\{g_i : i = 1, 2, \dots, m\}$  of  $G(s)$  may be called the corresponding set of open-loop characteristic gains, obtained from

$$\det[gI_m - G(s)] = 0 \quad (2.9)$$

The closed-loop characteristic frequencies  $\{s_i\}$  for a given value of  $g$  are, as is well known, associated with invariant subspaces in the state space  $\mathcal{X}$ . The open-loop characteristic gains  $\{g_i\}$  for a given value of  $s$  are associated with invariant subspaces in the input space  $\mathcal{U}$  (see Fig.3), whose physical interpretation will be discussed after stating the Generalized Nyquist Stability Criterion.

From the characteristic equations for  $G(s)$  and  $S(g)$ , namely

$$\Delta(g, s) \triangleq \det[gI_m - G(s)] = 0 \quad (2.10)$$

and

$$\nabla(s, g) \triangleq \det[sI_n - S(g)] = 0 \quad (2.11)$$

one obtains a pair of algebraic equations relating the complex variables  $s$  and  $g$ . These define a pair of algebraic functions [3]:

(i) a characteristic gain function  $g(s)$  which gives open-loop characteristic gain as a function of frequency; and

(ii) a characteristic frequency function  $s(g)$  which gives a closed-loop characteristic frequency as a function of gain.

The importance of these two algebraic functions lies in the fact that they are the natural means of generalising the concepts of Nyquist diagram and Root-Locus diagram to the multivariable case [3].

## 2.2 Generalised Nyquist Diagrams and the Generalised Nyquist Stability Criterion

The characteristic gain loci (generalised Nyquist diagrams) for the  $m$ -vector feedback loop with transmittance matrix  $G(s)$  are the loci in the complex gain plane which are traced out by the eigenvalues of  $G(s)$  as  $s$  traverses the so-called Nyquist  $D$ -contour in the complex frequency plane ( $s$ -plane). They can also be defined as the  $\pm 90^\circ$  constant-phase contours on the  $g$ -plane of the algebraic function  $s(g)$ . Their utility lies in their role in the following sort of generalisation of the Nyquist Stability Criterion [3], [4].

### Multivariable Nyquist Stability Criterion :

The multivariable Nyquist stability criterion relates the closed-loop stability of the configuration of Fig.5 to the characteristic gain loci for the loop-gain matrix  $G(s)H(s)$

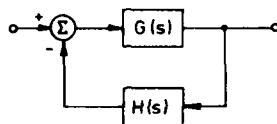


Figure 5

Suppose the system of Fig. 5 has no uncontrollable and/or unobservable modes whose corresponding characteristic frequencies lie in the right-half  $s$ -plane. Then this feedback configuration will be closed-loop stable if and only if the net sum of anti-clockwise encirclements of the critical point  $(-1+j0)$  by the set of characteristic gain loci of  $G(s)H(s)$  is equal to the total number of right-half-plane poles of  $G(s)$  and  $H(s)$ .

Proofs of multivariable versions of the Nyquist Stability Criterion have been given by Barman and Katzenelson [5] and MacFarlane and Postlethwaite [4], and Postlethwaite and MacFarlane [3] following earlier heuristic approaches by Bohn and Kasvand [6] and MacFarlane [7].

### 2.3 Physical Interpretation of Characteristic Gains

The idea of a characteristic frequency plays a fundamental role in applied dynamics. It is associated with an invariant subspace in a dynamical system's state space. This invariant subspace is associated in turn with a characteristic pattern of dynamical behaviour (a mode) and with an eigenvalue of a matrix mapping the state space into itself; this eigenvalue is the characteristic frequency associated with the mode.

A characteristic gain is associated with an invariant subspace in the input space to a feedback loop. This input space is a space of complex vectors. In physical terms one can set up a correspondence between the set of all complex vectors and the set of exponential vector waveforms of a given complex frequency. With reference to datum values of amplitude and phase for a given reference time, the components of a complex signal vector then define the relative amplitudes and phases of the various signal components which it represents. Now consider the situation shown in Fig. 6.

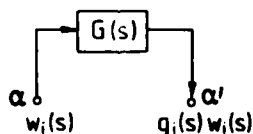


Figure 6

If the loop gain transmittance is  $G(s)$  and if the injected signal vector  $w_i(s)$  is an eigenvector of  $G(s)$  for the complex frequency  $s$ , then the returned signal will be  $g_i(s)w_i(s)$  where  $g_i(s)$  is the appropriate characteristic gain of  $G(s)$ . Hence in signal space this returned signal vector will lie in the same one-dimensional linear manifold as the input signal vector. Therefore one would expect, as stated in the Generalized Nyquist Stability Criterion, that characteristic frequencies associated with closing the loop through a unit negative gain must be such that the characteristic gain function satisfies the condition :

$$g(s) = -1.$$

So both characteristic frequencies and characteristic gains are associated with invariant subspaces. In the case of characteristic frequency this is a linear manifold along which state evolves in a state space. In the case of characteristic gain it is a linear manifold of signal vectors along which injected and returned signal vectors both lie in an input space.

#### 2.4 Multivariable Root Loci

The loci of open-loop characteristic gain with frequency are generalisations of classical Nyquist diagrams in the complex gain plane; there is a corresponding generalisation of the classical Evans' Root Locus diagram in the frequency plane. These are the characteristic frequency loci which are traced out by the eigenvalues of  $S(g)$  as  $g$  traverses the negative real axis in the gain plane. They can also be regarded as the  $180^\circ$  phase contours of  $g(s)$  on the frequency plane. The theory behind the multivariable root locus has been given by Postlethwaite and MacFarlane [3] and their use discussed by Retallack [8], Kouvaritakis and Shaked [9], MacFarlane, Kouvaritakis and Edmunds [10], Kouvaritakis and Edmunds [11], Kouvaritakis [12], and Owens [13]. Many examples of generalised Nyquist and Root-Locus diagrams are given in [2]; a complete review of the development of frequency-response methods is given in [1].

#### 2.5 Conformal Nature of Mapping Between Frequency and Gain

The algebraic functions  $g(s)$ , giving characteristic gain as a function of frequency, and  $s(g)$ , giving characteristic frequency as a function of gain contain essentially the same information. One is simply a "re-packaged" version of the other. One would thus expect there to be an exact structural relationship between the (generalised) Nyquist and (generalised) Root Locus diagrams for a given feedback loop. These structural relationships also reflect the fact that the mappings between  $s$ -plane and  $g$ -plane and vice-versa are conformal, so that appropriate angular relationships are preserved. In correlating the main features of the two forms of diagram, the following rule has been found useful.

**Locus-Crossing Rule**: Let  $D$  be a path in the  $s$ -plane and let  $D^1$  be its image under  $g(s)$  in the  $g$ -plane. Also let  $E$  be a path in the  $g$ -plane and let  $E^1$  be its image under  $s(g)$  in the  $s$ -plane. Then the following relationships hold.

(i) Each crossing of  $E$  by  $D^1$  corresponds to a related crossing of  $D$  by  $E^1$ , and vice versa.

(ii) If  $E^1$  crosses  $D$  from left to right, with respect to a given orientation for  $D$ , then  $E$  will cross  $D^1$  from left to right with respect to the induced orientation for  $D^1$ , and vice versa. (The given orientations for  $D$  and  $E$  are taken to induce appropriate orientations in  $D^1$  and  $E^1$ : that is a traversal of  $D$  by  $s$  in a positive orientation makes  $g(s)$  traverse  $D^1$  in a positive orientation, with a similar convention for  $E$  and  $E^1$ .)

(iii) The angle at which  $E^1$  crosses  $D$  is the same as the angle at which  $D^1$  crosses  $E$ .

This rule is illustrated in Fig.7, taking  $D$  as the positive imaginary axis in the  $s$ -plane and  $E$  as the negative real axis in the  $g$ -plane.

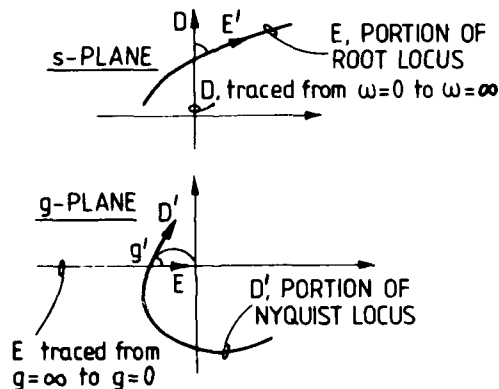


Figure 7

This shows that a crossing of the negative real axis in the gain plane (at  $g^1$  say) by a portion of the generalised Nyquist diagram corresponds to the migration of a closed-loop characteristic frequency from left to right (under negative feedback of amount  $(g^1)^{-1}$  applied equally in all loops).

### 3. PRINCIPAL GAINS AND PRINCIPAL PHASES

A natural way to look at the gain of an operator representing a multivariable system transmittance is in terms of the ratio of the norm of an output vector to the norm of a corresponding input vector. Thus if

$$y(s) = G(s) u(s) \quad (3.1)$$

one may define the vector gain (or simply the gain) of  $G(s)$  for input  $u(s)$  as

$$\text{gain } G|u = \frac{\|y\|}{\|u\|} \quad (3.2)$$

where  $\|\cdot\|$  denotes the standard Euclidean vector norm. In these terms one can see that the characteristic gains of  $G(s)$ , despite their useful role in closed-loop stability analysis, do not give an adequate description of the gain behaviour of an operator. For example the matrix transfer function

$$G(s) = \begin{bmatrix} 0 & 0 \\ \frac{s+1}{s+2} & 0 \end{bmatrix}$$

has characteristic gains which are identically zero for all values of  $s$ , yet it obviously does not have zero gain for all non-zero inputs. For this reason characteristic decompositions of an operator are not well suited to the discussion of the performance of feedback systems, and another form of operator decomposition is needed which is more suited to the accurate discussion of gain behaviour. Such a decomposition is found in terms of the singular values of an operator, and there is currently widespread interest in investigating the role of singular values in the analysis and design of multivariable feedback systems [14], [15], [16], [17], and [18].

It is natural to view the discarding of phase information which accompanies a singular value decomposition with some regret and thus to seek some way of retaining a measure of phase information. This can be done by making use of the polar decomposition of an operator using one half of the polar decomposition to provide gain information (as in the singular value decomposition) and the remaining part to give phase information. Since the term "singular phase" falls strangely on the ear, and since the term gain is more immediately relevant to physical concepts than the (admittedly well-established) mathematical term singular, one can adopt the terms principal gain and principal phase in this characterization of operators [17], [18].

Any square complex matrix  $G$  can be represented in the forms

$$G = UH_R \quad (3.3)$$

$$G = H_L U \quad (3.4)$$

where  $U$  is unitary and  $H_R, H_L$  are positive semi-definite Hermitian matrices.  $H_R$  and  $H_L$  are often called the right and left moduli of  $G$  and are given by

$$H_R = \sqrt{G^* G} \quad (3.5)$$

$$H_L = \sqrt{G G^*} \quad (3.6)$$

where  $G^*$  is the complex conjugate transpose of  $G$ .  $U$  is then uniquely defined by (3.3) or (3.4) in case  $G$  is non-singular. Using these polar decompositions one can make the following definitions:

(i) the principal gains of  $G$  are the eigenvalues of the Hermitian part of its polar decomposition; and

(ii) the principal phases of  $G$  are the arguments of the eigenvalues of the unitary part of its polar decomposition [18].

Since the non-zero eigenvalues of  $G^* G$  and  $G G^*$  are the same either the left or right Hermitian factor may be used in determining the principal gains. Furthermore the principal gains are the same as the singular values of  $G$ . These principal gains and principal phases are related to the characteristic gains in the following way [18]:

(i) The magnitudes of the characteristic gains of a complex matrix  $G$  are bounded above and below by the corresponding maximum and minimum principal gains.

(ii) If the principal phases of a complex matrix  $G$  have a spread of less than  $\pi$  radians then the arguments of the characteristic gains of  $G$  are bounded above and below by the maximum and minimum principal phases.

If the zeros of  $G(s)$  are defined in the usual way in terms of its Smith-McMillan decomposition [19] then one can show that the vector gain of  $G(s)$  vanishes for  $s = z$  if and only if  $z$  is a zero of  $G(s)$  [17].

Principal gain decompositions are useful in analysing the performance of feedback systems. For example Daniel [20] has used them to show how in a regulator the use of rank-deficient feedback can lead to disturbance amplification.

Using a combination of principal gains and principal phases one can develop a Nyquist-type sufficient stability criterion which can be used to characterize the robustness of closed-loop stability when the system model is subjected to a linear perturbation (of either multiplicative or additive form) at any point in the feedback loop [18]. Such an approach gives results which are much less conservative than those obtainable using the Small Gain Theorem (since this neglects phase aspects of the problem).

For an open-loop  $m \times m$  gain matrix  $G(s)$  it is clear from properties (i) and (ii) above that for any complex frequency  $s = j\omega$  a curvilinear rectangle can be drawn from the maximum and minimum principal values, within which the corresponding  $m$  values of the characteristic gain loci must lie. Should the principal phases have a spread greater than  $\pi$  then the characteristic gains lie inside an annular region determined by the maximum and minimum principal gains. If these rectangles or annular regions are constructed for values of  $s$  around the Nyquist D-contour then a region will be outlined in the gain plane, which we shall call the principal region, within which the characteristic gain loci must lie as shown in Fig.8. Application of the generalized Nyquist stability criterion to this region leads to the stability criterion of section 4 below.

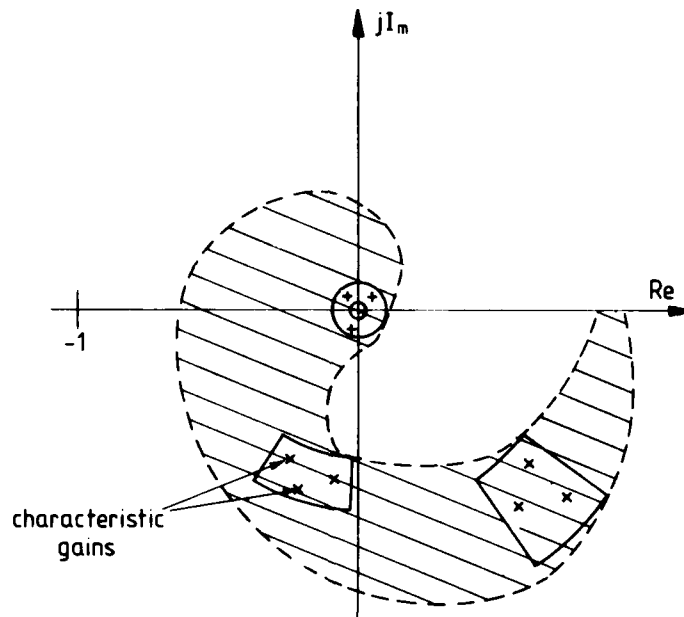


Figure 8 Principal region

### 3.1 Relationships between the characteristic gains and the principal gains and phases of a complex matrix.

In this sub-section two important theorems are presented which relate the characteristic gains to the principal gains and phases of a complex matrix,  $T$ . The theorems are crucial to the stability criterion of the next section.

**Theorem 1** The magnitudes of the characteristic gains of a complex matrix  $T$  are bounded above and below by the corresponding maximum and minimum principal gains. Although this theorem is well known (see for example [17] or [21]) it is included in the main text because of its importance in the development of the major results of this paper and also to complement theorem 2 which is unfamiliar.

Proof

$$\text{Let } T^* T m_i = \gamma_i^2 m_i \quad i = 1, 2, \dots, m \quad (3.7)$$

where  $T$  is a complex  $m \times m$  matrix, the  $\{m_i\}$  are an orthonormal set of eigenvectors for  $T^* T$ , and the  $\{\gamma_i\}$  are the principal gains of  $T$ , numbered so that  $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_m$ .

$$\text{Also let } T v_i = \lambda_i v_i \quad i = 1, 2, \dots, m \quad (3.8)$$

where the  $\{v_i\}$  are a set of eigenvectors for  $T$  and the  $\{\lambda_i\}$  are the corresponding eigenvalues.

Any "input" vector  $u$  can be represented in the form

$$u = \sum_{i=1}^m c_i m_i \quad c_i \in \mathbb{C} \quad (3.9)$$

and therefore

$$\frac{\|Tu\|_2^2}{\|u\|_2^2} = \frac{u^* T^* T u}{u^* u} = \frac{\sum_{i=1}^m |c_i|^2 \gamma_i^2}{\sum_{i=1}^m |c_i|^2} = \sum_{i=1}^m f_i \gamma_i^2 \quad (3.10)$$

where  $f_i = \frac{|c_i|^2}{\sum_{j=1}^m |c_j|^2}$ ,  $f_i \geq 0$ , and  $\sum_{i=1}^m f_i = 1$ .

That is, the ratio  $\frac{\|Tu\|_2^2}{\|u\|_2^2}$  is a convex combination [21], of the  $\{\gamma_i^2\}$  which

implies that

$$\gamma_1 \leq \frac{\|Tu\|_2}{\|u\|_2} \leq \gamma_m \quad \text{for all } u. \quad (3.11)$$

In particular for  $u = v_i$ ,  $\frac{\|Tv_i\|_2}{\|v_i\|_2} = |\lambda_i|$ , and therefore

$$\gamma_1 \leq |\lambda_i| \leq \gamma_m \quad (3.12)$$

**Theorem 2.** If the principal phases of a complex matrix  $T$  have a spread of less than 180 degrees then the arguments of the characteristic gains of  $T$  are bounded above and below by the maximum and minimum principal phases.

This theorem is a special case of a more general result by Ali R. Amir-Moëz and Alfred Horn [22], theorem 4, page 745].

Proof

Let  $T$  have the polar decomposition

$$T = U H_R \quad (3.13a)$$

$$\text{let } T v = \lambda v \quad (3.13b)$$

where  $v$  is an eigenvector of  $T$  and  $\lambda$  a corresponding eigenvalue, and let

$$U z_i = \rho_i z_i \quad i = 1, 2, \dots, m \quad (3.14)$$

where the  $\{z_i\}$  are an orthonormal set of eigenvectors for  $U$  and the  $\{\rho_i\}$  are the corresponding eigenvalues, the arguments of which are the principal phases of  $T$ . Then

$$\begin{aligned} \lambda \langle H_R v, v \rangle &= \langle H_R v, T v \rangle = v^* H_R^* U H_R v \\ &= v^* H_R^* \left[ \sum_{i=1}^m \rho_i z_i z_i^* \right] H_R v \\ &= \sum_{i=1}^m \rho_i \langle H_R v, z_i \rangle \langle z_i, H_R v \rangle \\ &= \sum_{i=1}^m \rho_i |\langle H_R v, z_i \rangle|^2 \end{aligned} \quad (3.15)$$

$T$  has implicitly been assumed nonsingular and consequently so has  $H_p$  which is also Hermitian by definition. It follows therefore that  $\langle H_p v, v \rangle$  is both real and greater than zero which, with (3.15), implies that  $\lambda$  lies in the convex cone generated by the  $\{v_i\}$ . If the cone is not convex, that is the principal phases have a spread greater than 180 degrees, we cannot say anything about the argument of  $\lambda$ . A convex cone [15] is a set of numbers in the complex plane which are closed under linear combinations with nonnegative coefficients. If we consider drawing an arc in the complex plane, centre the origin, in an anti-clockwise direction, then the angle at which the arc enters the convex cone is referred to as the minimum principal phase and the angle at which the arc leaves the cone is the maximum principal phase. We have therefore shown that the arguments of the characteristic gains of  $T$  are bounded above and below by the corresponding maximum and minimum principal phases.

#### 4. A SUFFICIENT STABILITY CRITERION

In the generalised Nyquist stability criterion [5], [4] and [23], the stability of a linear multivariable feedback configuration, see Fig. 9, is determined from the characteristic gain loci which are a polar plot of the eigenvalues of the open-loop gain matrix  $G(s)$ , evaluated as  $s$  traverses the standard Nyquist D-contour. This criterion can be stated as follows:

the closed-loop system is stable if and only if the number of anti-clockwise encirclements of the critical point  $(-\frac{1}{k} + j0)$  by the characteristic gain loci is equal to the number of open-loop unstable poles.

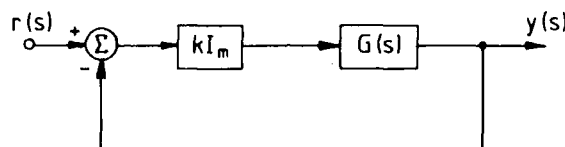


Figure 9 Multivariable feedback configuration

It is now shown how, using theorems 1 and 2, a sufficient Nyquist-type stability criterion can be obtained in terms of the principal gains and phases of  $G(j\omega)$ .

For an open-loop gain matrix  $G(s)$  it is clear from theorems 1 and 2 that for any complex frequency  $s=j\omega$  a curvilinear rectangle can be drawn from the maximum and minimum principal values, within which the corresponding  $m$  values of the characteristic gain loci (generalised Nyquist diagrams) lie. Should the principal phases not form a convex cone then the characteristic gains lie inside an annular region determined by the maximum and minimum principal gains. If these rectangles or annular regions are constructed for values of  $s$  around the Nyquist D-contour then a region will be outlined in the gain plane, which we shall call the principal region, within which the characteristic gain loci must lie; see Fig. 8. Application of the generalised Nyquist stability criterion to this region now leads to the following stability criterion:

the closed-loop system is stable if  $m$  times the number of anti-clockwise encirclements of the critical point  $(-\frac{1}{k} + j0)$  by the principal region is equal to the number of open-loop unstable poles.

It also follows that:

the closed-loop system is unstable if  $m$  times the number of anti-clockwise encirclements of the critical point  $(-\frac{1}{k} + j0)$  by the principal region is not equal to the number of open-loop unstable poles.

When the critical point lies inside the principal region neither stability nor instability can be inferred.

Since the above are only sufficient conditions for stability or instability, as straightforward stability tests they are inferior to the generalised Nyquist stability criterion. However, the generalised Nyquist stability criterion only determines stability with respect to a single gain common to all the loops and as such does not reliably characterize the robustness of the stability property when the system is subjected to arbitrary perturbations. In the next section it is shown how a new sufficient Nyquist-type stability criterion can be used to characterize the robustness of a linear multivariable design when it is subjected to linear perturbations.

## 5. ROBUST STABILITY

One of the major reasons for using feedback control as opposed to open-loop is the presence of model uncertainties and hence the desire to keep the system stable under large parameter variations. However, the mere presence of feedback is not sufficient to guarantee the robust stability of a system [24]. Research in this area has recently [25], [26a], and [27] centred around finding conditions for which a feedback system will remain stable when the open-loop gain matrix  $G(s)$  is subjected to a multiplicative or additive perturbation  $\Delta G(s)$ . It has been shown that the feedback configurations of figures 10 and 11 remain stable when  $G(s)$  is stable and

$$\|\Delta G(j\omega)\| < \frac{1}{\|[I + G(j\omega)]^{-1}\|} \quad \text{for all } \omega \quad (5.1)$$

in the multiplicative case, and

$$\|\Delta G(j\omega)\| < \frac{1}{\|[I + G(j\omega)]^{-1}\|} \quad \text{for all } \omega \quad (5.2)$$

in the additive case. These relationships are most easily derived from a simple application of the small gain theorem, [27], [28] and [53], after first rearranging the block diagrams. If the spectral norm  $\|\cdot\|_2$  is taken, then the robustness of each configuration is characterized by the maximum principal gain (singular value) of the appropriate frequency response matrix.

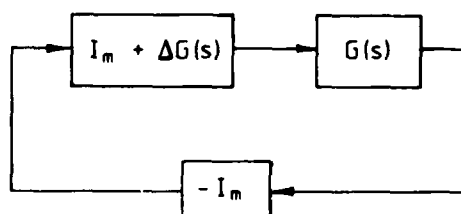


Figure 10 Feedback configuration with multiplicative perturbation

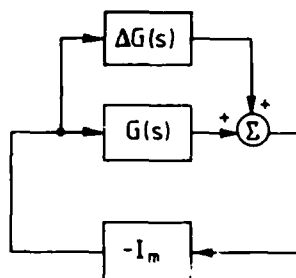


Figure 11 Feedback configuration with additive perturbation

We shall now examine how the new stability criterion of section 4 might be used to characterize robustness. Motivated by the inclusion of phase information, the aim is to obtain less conservative bounds on a linear perturbation than those furnished by the small gain theorem.

### 5.1 Gain and phase margins

To use the new stability criterion in the context of robustness we firstly rearrange figures 10 and 11 to those in figures 12 and 13 respectively, and make the observation that the system in figure 10 is stable if and only if the system of figure 12 is stable, and likewise for figures 11 and 13. The arguments that follow are



essentially the same for both the multiplicative and additive case and so for simplicity we will consider only the multiplicative case.

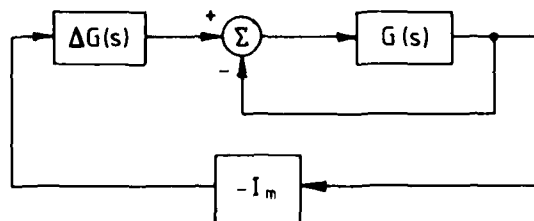


Figure 12 Rearranged feedback configuration with multiplicative perturbation

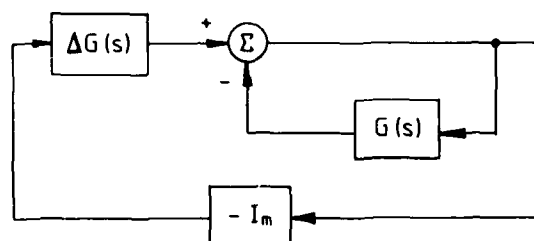


Figure 13 Rearranged feedback configuration with additive perturbation

We now construct the principal region for  $[I + G(j\omega)^{-1}]^{-1}$  and ask the question: Can we deduce from the principal gains and phases of  $\Delta G(j\omega)$  a deformed principal region within which the eigenloci (characteristic gain loci) of  $\Delta G(j\omega)[I + G(j\omega)^{-1}]^{-1}$  must lie? If this is so then we will be able to specify conditions on  $\Delta G(j\omega)$  for which the perturbed system will remain stable. The answer is yes, and the conditions on  $\Delta G(j\omega)$  for which the system remains stable are summarised in the following theorems, after first introducing some notation.

Let  $[I + G(j\omega)^{-1}]^{-1}$  have principal gains and phases

$$\alpha_1(\omega) < \alpha_2(\omega) < \dots < \alpha_m(\omega) \quad (5.3)$$

and

$$\theta_1(\omega) < \theta_2(\omega) < \dots < \theta_m(\omega) \quad (5.4)$$

respectively, where the numbering of the  $\{\theta_i(\omega)\}$  is such that the notions of maximum and minimum phases are as defined at the end of section 3; it is assumed that the  $\{\theta_i(\omega)\}$  have a spread of less than  $\pi$ . Similarly let the principal gains and phases of  $\Delta G(j\omega)$  be  $\delta_1(\omega) < \delta_2(\omega) < \dots < \delta_m(\omega)$  and

$$\varepsilon_1(\omega) < \varepsilon_2(\omega) < \dots < \varepsilon_m(\omega) \quad (5.6)$$

respectively, where the phases  $\{\varepsilon_i(\omega)\}$  can be thought of as negative or positive arguments corresponding to unmodelled phase lag or advance characteristics. Also let the condition numbers [26b] for  $[I + G(j\omega)^{-1}]^{-1}$  and  $\Delta G(j\omega)$ , using the  $\ell_2$ -induced norm, be  $c_1(\omega)$  and  $c_2(\omega)$ , respectively, so that

$$c_1(\omega) \triangleq \frac{\alpha_m(\omega)}{\alpha_1(\omega)} \quad \text{and} \quad c_2(\omega) \triangleq \frac{\delta_m(\omega)}{\delta_1(\omega)} \quad (5.7)$$

and also define

$$\psi_m(\omega) \triangleq \tan^{-1} \left[ \frac{|c_1(\omega) - 1| c_2(\omega)}{1 - |c_1(\omega) - 1| c_2(\omega)} \right] \quad (5.8)$$

which will be referred to as a phase modifier.

We can now state and prove the following theorems.

**Theorem 3 (Small Gain Theorem):** The closed-loop system remains stable under a multiplicative perturbation  $\Delta G(s)$  if

- a)  $\Delta G(s)$  is stable and
- b)  $\delta_m(\omega) \alpha_m(\omega) < 1$  for all  $\omega$ .

This is exactly the small gain theorem introduced by Sandberg [27] and Zames [28] for a more general class of systems. The result is often conservative, especially in our case where we are considering linear systems, but the conservativeness can be reduced if we consider the principal phases as described in the next theorem.

**Theorem 4 (Small Phase Theorem):** The closed-loop system remains stable under a multiplicative perturbation  $\Delta G(s)$  if

- a)  $\Delta G(s)$  is stable,
- b)  $\{\theta_1(\omega) + \epsilon_j(\omega) : j=1, \dots, m\}$  have a spread of less than  $\pi$  for all  $\omega$ ,
- c)  $|c_1(\omega) - 1|c_2(\omega) < 1$  for all  $\omega$ ,
- d)  $\epsilon_1(\omega) + \theta_1(\omega) - \psi_m(\omega) > -\pi$  for all  $\omega$ , and
- e)  $\epsilon_m(\omega) + \theta_m(\omega) + \psi_m(\omega) < \pi$  for all  $\omega$ .

Note that conditions d) and e) are analogous to the phase margin concept used in the analysis of single-input-single-output systems modulo the phase modification  $\psi_m(\omega)$ .

In general, Theorem 4 is more applicable at low frequencies when  $\theta_1(\omega)$  is a small lag. At higher frequencies, when  $\theta_1(\omega)$  approaches  $-\pi$  and the principal region is in the vicinity of the "-1" point, then Theorem 3 is more useful. Consequently, for a more practical characterization of the robustness of the stability property over the whole frequency range, Theorems 3 and 4 are combined, as in Theorem 5 below.

**Theorem 5:** The closed-loop system remains stable under a multiplicative perturbation  $\Delta G(s)$  if, for some frequency  $\omega_b$ ,

- a) the conditions of Theorem 4 are satisfied in the frequency range  $[0, \omega_b]$  and
- b) the conditions of Theorem 3 are satisfied in the frequency range  $[\omega_b, \infty]$ .

The proofs of Theorems 3 and 4 are given in [18]; essentially they are based on the following ideas. In Theorem 3 we restrict the moduli of the characteristic gain loci (generalised Nyquist diagrams) to values less than 1, so we can have no encirclements of the critical point -1, and hence closed-loop stability regardless of the phases involved. In Theorem 4 we prevent the characteristic gain loci from crossing the negative real axis by restricting their arguments, so again we can have no encirclements of the critical point, and hence closed-loop stability regardless of the associated moduli.

In Theorem 5 we prevent the characteristic gain loci from encircling the critical point by restricting their arguments in the frequency range  $[0, \omega_b]$  and restricting their moduli in the frequency range  $[\omega_b, \infty]$ . The proofs of Theorems 3 and 4 therefore show that Theorem 5 holds, and hence no formal proof of Theorem 5 is given.

**Interpretation of Theorems 3, 4 and 5:** Theorem 3 tells us that stability is maintained for a stable linear perturbation  $\Delta G(s)$  if the spectral norm of the perturbation frequency response is always less than a value determined from the system model. Therefore, in the polar decomposition for  $\Delta G(j\omega)$  there is an upper limit on the maximum principal gain at each frequency, but no restriction at all on the principal phases.

Theorem 4 shows us how extra gain can be accommodated in the perturbation providing certain phase requirements are satisfied. Conditions b) and c) are mathematical requirements for the definition of the phase modifier  $\psi_m(\omega)$  and the development of the phase conditions d) and e); see [18]. More precisely, condition b) implies that at any frequency, the two sets of principal phases, one for  $\Delta G(j\omega)$  and the other for  $[I + G(j\omega)^{-1}]^{-1}$ , each lie in a convex cone, that is, have a spread less than  $\pi$ , and also that the "sum" of these convex cones is convex. The sum of two cones is here defined as the cone formed from the products of all complex numbers in one cone with those in the other. The term "sum" is used because the sum of two convex cones is clearly defined by summing the principal phases of one cone with those of the other. Condition c) restricts the condition numbers of  $c_1(\omega)$  and  $c_2(\omega)$ ;  $c_1(\omega)$  is required to be less than 2 and  $c_2(\omega)$  less than  $1/|c_1(\omega) - 1|$ . In practice, for a design which aims at accurate tracking through approximate diagonalization,  $c_1(\omega)$  will be about 1, allowing a large value for  $c_2(\omega)$ , the condition number of the perturbation. Note, however, that a large value of  $c_2(\omega)$  corresponds to a large value of  $\psi_m(\omega)$  which from condition d) imposes a greater restriction on  $\epsilon_1(\omega)$  as a phase lag in the perturbation. As  $c_2(\omega)$  is reduced, the phase lag in the perturbation can be increased, that is, condition number can be traded for phase and vice versa. It should be noted that  $c_2(\omega) < \infty$  implies that  $\Delta G(j\omega)$  is nonsingular.

## 5.2 Automatic Scaling

Mees and Edmunds [29] have recently developed and implemented on the Cambridge multivariable design package an algorithm which effectively finds a real diagonal matrix  $S$  such that  $S^{-1}T(j\omega)S$  is as near diagonal as possible over the whole frequency range. This automatic scaling technique can be used in the study of stability with respect to a perturbation  $\Delta G(s)$ , as is now explained for the "multiplicative" case.

Instead of constructing the principal region for  $[I + G(j\omega)^{-1}]^{-1}$ , let us consider the principal region for  $S^{-1}[I + G(j\omega)^{-1}]^{-1}S$ . This principal region will, in general, have a narrower band, and since Fig 12 can be redrawn as shown in Fig.14 we see that it predicts stability with respect to the perturbation  $S^{-1}\Delta G(s)S$ . The scaling matrix  $S$  therefore tells us which elements of  $\Delta G(s)$  are, relatively speaking, the most sensitive.

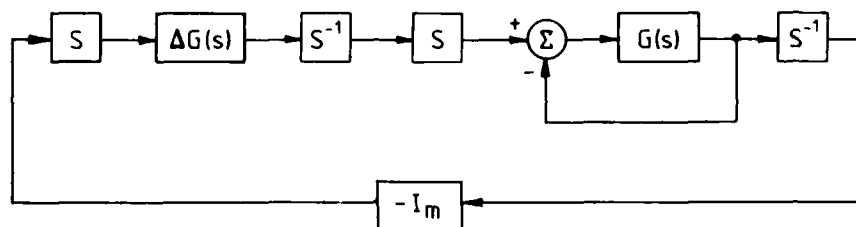


Figure 14 Feedback configuration with scaling

## 5.3 Additive Perturbations

As stated earlier, one can just as easily consider the "additive" case, the only essential difference being that the principal region for  $[I + G(j\omega)]^{-1}$ , assuming integral action, is in the neighbourhood of the origin at low frequencies and in the neighbourhood of one at high frequencies. In the additive case, therefore, the small gain theorem is more applicable at low frequencies and the small phase theorem is more applicable at high frequencies, in contrast to the multiplicative case. In general, a linear perturbation, multiplicative or additive, can be considered at any point in a feedback system by constructing the appropriate principal region.

## 6. ZEROS

The characteristic gain function  $g(s)$  has a set of poles and zeros, and its related generalised Nyquist and Root-Locus diagrams have distinctive patterns of asymptotic behaviour. It is of great value to relate all these to the state-space model parameters of the feedback loop dynamics. The poles are simply a subset of the eigenvalues of an appropriate A-matrix if the state-space model involved is not fully controllable; normally they will be the full set of A-matrix eigenvalues. This section looks at the relationship between the zeros of  $g(s)$  and the state-space structure, and Section 8 looks at asymptote behaviour.

In physical terms zeros are associated with the vanishing of vector gain; that is with the existence of non-zero input exponential signal vectors which result in zero output [30]. The discussion of zeros in the literature is confusing because one can talk about zeros in connection with various objects:  $\Sigma(A,B,C,D)$ , or  $G(s)$ , or  $g(s)$  for example. The object  $\Sigma(A,B,C,D)$  has associated with it a larger set of zeros than the object  $G(s)$  because in passing from the representation  $\Sigma(A,B,C,D)$  to the representation  $G(s)$  one discards phenomena (uncontrollability and unobservability) which can be discussed in terms of zeros (the so-called decoupling zeros [31]). In turn the object  $G(s)$  has a larger set of zeros than the object  $g(s)$  because vanishing of all characteristic gains does not necessarily imply the vanishing of the vector gain for  $G(s)$ . For the purposes of this summary we will take the system representations being discussed to be completely controllable and completely observable, and the feedback loop transmission zeros to be those values of  $s$  obtained by considering the consequences of putting  $g = 0$  in a specific way. A useful physical interpretation of a zero then comes from the following result [19].

**Transmission-blocking theorem:** For a system  $\Sigma(A,B,C,D)$  having a number of inputs less than or equal to the number of outputs, necessary and sufficient conditions for an input of the form

$$u(t) = l(t)e^{zt}u_z \quad (6.1)$$

to result in a state-space trajectory

$$x(t) = l(t)e^{zt}x_z \quad \text{for } t > 0 \quad (6.2)$$

and an output

$$y(t) \equiv 0 \text{ for } t > 0 \quad (6.3)$$

are that a complex frequency  $z$  and a complex vector  $u_z$  exist such that

$$\begin{bmatrix} zI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = 0 \quad (6.4)$$

(Here  $l(t) = 0$  for  $t \leq 0$  and  $l(t) = 1$  for  $t > 0$ .)

Hence the zeros defined in this way are the roots of the invariant factors of an appropriate pencil. The corresponding vectors  $u_z$  and  $x_z$  are called the zero directions in the input and state spaces respectively [19]

Zero Pencil: From (6.4) we have that, for the case when  $D = 0$ ,

$$(zI - A)x_z = Bu_z \quad (6.5)$$

$$Cx_z = 0 \quad (6.6)$$

Zero directions in the state space naturally lie in the kernel of  $C$  so that

$$x_z = Mv_z \quad (6.7)$$

where  $M$  is a basis matrix representation of  $\text{Ker } C$  and  $v_z$  is an appropriate vector of constants. Substitution of (6.7) into (6.5) and premultiplication by the full-rank transformation matrix  $\begin{bmatrix} N \\ B^+ \end{bmatrix}$  where  $N$  is a full-rank left annihilator of  $B$  (i.e.  $NB=0$ ) and  $B^+$  is a left inverse of  $B$  (i.e.  $B^+B=I_k$ ) yields

$$(zNM - NAM)v_z = 0 \quad (6.8)$$

$$u_z = B^+(zI - A)Mv_z \quad (6.9)$$

The object  $(sNM - NAM)$  is called the zero pencil [32], [33] for the system  $\Sigma(A, B, C)$ . The zeros are the roots of the invariant factors of the zero pencil, and this fact can be used to exhibit the geometric characterization of zeros in a very succinct way. To do this it is useful to use Sastry and Desoer's notion of restricting an operator in domain and range [34].

#### 6.1 Restriction of an operator in domain and range

Given a linear map  $A$  from  $\mathbb{C}^n$  to  $\mathbb{C}^n$  and two subspaces  $\mathcal{J}_1, \mathcal{J}_2$  of  $\mathbb{C}^n$ , where  $\mathbb{C}^n$  is the space of  $n$ -dimensional complex vectors, then the restriction of  $A$  to  $\mathcal{J}_1$  in the domain and  $\mathcal{J}_2$  in the range is defined to be the linear map which associates with  $x_1 \in \mathcal{J}_1$  the orthogonal projection of  $Ax_1$  on to  $\mathcal{J}_2$ . This restricted linear map (and also its matrix representation) will be denoted by

$$A|_{\mathcal{J}_1 \rightarrow \mathcal{J}_2} \stackrel{\Delta}{=} A^v \quad (6.10)$$

If the columns of  $S_1$  and  $S_2$  respectively form orthonormal bases for  $\mathcal{J}_1$  and  $\mathcal{J}_2$  then the matrix representation of the restricted operator is given by

$$A^v = S_2^* A S_1 \quad (6.11)$$

where  $*$  denotes complex conjugate transposition. With a mild abuse of language one then calls the roots of

$$\det [S_2^* S_1 \lambda - S_2^* A S_1] = 0 \quad (6.12)$$

the spectrum of the restricted operator  $A^v$ .

#### 6.2 Spectral characterization of zeros

Suppose we choose  $N^*$  and  $M$  so that their columns are orthonormal bases for  $\text{Ker } B^*$  and  $\text{Ker } C$  respectively. Then

$$NM = I_n \Big|_{\text{Ker } C \rightarrow \text{Ker } B^*} \quad (6.13)$$

and

$$NAM = A \Big|_{\text{Ker } C \rightarrow \text{Ker } B^*} \quad (6.14)$$

and one may say that :

the finite zeros of  $\Sigma(A, B, C)$  are the spectrum of the restriction of  $A$  to  $\text{Ker } C$  in domain and  $\text{Ker } B^*$  in range.

## 6.3 Interpretation of zeros and zero directions in terms of intersecting manifolds.

$$\text{Let } J(s) \triangleq (sI - A)^{-1}B \quad (6.15)$$

and for a given value of complex frequency  $s$  consider the subspace  $\text{Im}J(s) \subset \mathbb{X}$ . If  $B$  is of full rank then any vector  $x_s \in \text{Im}J(s)$  can be made a closed-loop eigenvector of  $\Sigma(A, B, C)$  under state-feedback; also if  $x_s$  is a closed-loop eigenvector of  $\Sigma(A, B, C)$  under state-feedback it must lie in  $\text{Im}J(s)$ . Now it is a known property of closed-loop characteristic frequencies that, under feedback of unbounded norm, an appropriate number of closed-loop characteristic frequencies migrate to the finite zeros of the loop transmission matrix [35]. Since for arbitrarily high feedback gain, the relevant closed-loop poles will coincide with the zeros, an appropriate number of closed-loop modes will have become unobservable and their corresponding closed-loop eigenvectors will thus lie in  $\text{Ker } C$ . Hence one would expect that the zero direction for a finite zero  $z$  would be associated with the intersection of  $\text{Im}J(z)$  with  $\text{Ker } C$ . (Here  $\text{Im}$  denotes image.)

6.4 Finite zeros when  $D$  is non-zero.

Suppose the direct-coupling operator  $D$  for  $\Sigma(A, B, C, D)$  having  $m$  inputs and  $m$  outputs has nullity  $d_0$ . Then  $\Sigma(A, B, C, D)$  has at most  $(n - d_0)$  finite zeros and at least  $d_0$  root-locus asymptotes going to infinity (to infinite zeros) [10]. Let  $D$  have a characteristic decomposition

$$D = \begin{bmatrix} U_0 & M_0 \end{bmatrix} \begin{bmatrix} J_0 & 0 \\ 0 & O_{d_0} \end{bmatrix} \begin{bmatrix} V_0 \\ N_0 \end{bmatrix} \quad (6.16)$$

where  $[U_0 \ M_0]$  and  $[V_0^t \ N_0^t]^t$  are the eigenvector and dual eigenvector matrices for  $D$  and  $J_0$  is the Jordan block associated with the non-zero eigenvalues of  $D$ . Further suppose that there is a complete set of eigenvectors spanning the null space of  $D$ . Define  $A_0, B_0, C_0$  by

$$A_0 = A - BU_0J_0^{-1}V_0^t C \quad (6.17)$$

$$B_0 = BM. \quad (6.18)$$

$$C_0 = N_0 C \quad (6.19)$$

Then the finite zeros and root locus asymptotes of  $\Sigma(A, B, C, D)$  are those of  $\Sigma(A_0, B_0, C_0)$ .

In case  $D$  has full rank one has that there are  $n$  finite zeros of  $\Sigma(A, B, C, D)$  which are given by  $\sigma(A - BD^{-1}C)$ , the spectrum of the matrix  $(A - BD^{-1}C)$ .

## 7. BILINEAR TRANSFORMATION OF FREQUENCY AND GAIN VARIABLES

If  $\Sigma(A, B, C, D)$  having  $m$  inputs and  $m$  outputs has a  $D$  which is non-singular then, as noted in Section 6.4, it will have  $n$  finite zeros given by  $\sigma(A - BD^{-1}C)$ .

Given a bilinear transformation

$$p = \frac{as + b}{cs + d} \quad (7.1)$$

$$s = \frac{-dp + b}{cp - a} \quad (7.2)$$

on the complex frequency it has been shown by Edmunds [36] that it is possible to find a correspondingly transformed system  $\Sigma(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  such that

$$\tilde{G}(p) = G(s) \quad (7.3)$$

Thus if the original system has a zero at  $z$  the transformed system will have a zero at a location given by substituting  $z$  in (7.1). One can therefore choose an appropriate bilinear transformation to get a  $\tilde{D}$  of full rank, calculate the zeros as  $\sigma(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})$  and transform back to find the complete set of zeros (finite and infinite) of the original system. The bilinear mapping is a conformal map between the  $s$ -plane and the  $p$ -plane for all points except  $p = a/c$  which gets sent to infinity on the  $s$ -plane and  $s = -d/c$  which gets sent to infinity on the  $p$ -plane. It is assumed that  $(ab - bc) \neq 0$  and that the original system has no pole at  $-d/c$ . The set of transformed system matrices is given by

$$\tilde{A} = (cA + dI)^{-1}(aA + bI) \quad (7.4)$$

$$\tilde{B} = (cA + dI)^{-1}B \quad (7.5)$$

$$\tilde{C} = (ad - bc)C(cA + dI)^{-1} \quad (7.6)$$

$$\tilde{D} = D - cC(cA + dI)^{-1}B \quad (7.7)$$

A similar bilinear transformation can be carried out on the complex gain variable

$$q = \frac{a g + b}{c g + d} \quad (7.8)$$

$$g = \frac{d q + b}{c q - a} \quad (7.9)$$

The correspondingly transformed system  $\Sigma(A^1, B^1, C^1, D^1)$  is such that

$$S^1(q) = S(g) \quad (7.10)$$

and has state-space parameters given by

$$A^1 = A - cB(cD + dI)^{-1}C \quad (7.11)$$

$$B^1 = (ab - bc)B(cD + dI)^{-1} \quad (7.12)$$

$$C^1 = (acD + dI)^{-1}C \quad (7.13)$$

$$D^1 = (cD + dI)^{-1}(aA + bI) \quad (7.14)$$

## 8. GEOMETRIC THEORY OF ROOT LOCUS AND NYQUIST DIAGRAMS

It has been noted that the finite zeros of a vector feedback loop transmittance can be characterized in terms of the spectrum of the restriction of its A-matrix in domain and range. The idea of using the spectrum of a restricted operator in connection with the Markov Parameters of a loop transfer function matrix leads to a geometric treatment of the high-gain asymptotic behaviour of generalised Root-Locus diagrams and the high-frequency asymptotic behaviour of generalised Nyquist diagrams.

Let the transfer function matrix for a strictly proper (i.e.  $D = 0$ ) feedback loop be expanded in a Taylor series about  $s = \infty$  as

$$G(s) = \frac{G_1}{s} + \frac{G_2}{s^2} + \dots \quad (8.1)$$

where  $\{G_i : i=1,2,\dots\}$  are the Markov Parameters

$$G_1 = CB \quad G_k = CA^{k-1}B \quad k = 2,3,\dots \quad (8.2)$$

The use of Markov Parameters to relate the main features of generalised Root-Locus diagrams to state-space model parameters has been discussed by a number of investigators [37], [38], [39], [40][34]. The approach and notation used here follows Sastry and Desoer [34], who give a good discussion of the assumptions involved in a detailed analysis of this sort.

Let a sequence of restricted operators be defined as follows :

$$G_k^v = G_k \Big|_{\text{Ker } G_{k-1}^v \rightarrow \text{Ker } \begin{bmatrix} v \\ G_{k-1} \end{bmatrix}^*} \quad (8.3)$$

and take  $G_1^v = G_1$ .

Let  $d_i$  be the nullity (rank defect) of  $G_i^v$ .

Then the high-gain asymptotic behaviour of the various branches of the generalised root-locus diagram can be determined as follows.

1st order branches:  $s_{i,1} \approx -\frac{\lambda_{i,1}}{g} \quad i=1,2,\dots,(m-d_1)$

where  $\lambda_{i,1} \in \sigma[G_1^v] \setminus \{0\}$  (8.4)

and  $g \rightarrow 0$

The collection of symbols  $\sigma[G_1^v] \setminus \{0\}$  is to be read as "belongs to the non-zero spectrum of  $G_1^v$ ."

2nd order branches:  $s_{i,2} \approx -\frac{\lambda_{i,2}}{g} \quad i=1,2,\dots,(d_2-d_1)$

where  $\lambda_{i,2} \in \sigma[G_2^v] \setminus \{0\}$  (8.5)

and  $g \rightarrow 0$

and for the  $k^{\text{th}}$  order branches

$$s_{i,k} \approx -\frac{\lambda_{i,k}}{g} \quad i = 1,2,\dots,(d_k - d_{k-1})$$

where  $\lambda_{i,k} \in \sigma[G_k^v] \setminus \{0\}$  (8.6)

and  $g \rightarrow 0$

On invoking the correspondence between the high-frequency asymptotes for the generalised Nyquist diagrams and the high-gain asymptotes for the generalised Root-Locus diagrams one then gets the following description of the high-frequency branches of the generalised Nyquist diagrams.

$$\text{1st order branches: } g_{i,1} \approx - \frac{\lambda_{i,1}}{j\omega} \quad (8.7)$$

$$\text{2nd order branches: } g_{i,2} \approx - \frac{\lambda_{i,2}}{(j\omega)^2} \quad (8.8)$$

$$\text{kth order branches: } g_{i,k} \approx - \frac{\lambda_{i,k}}{(j\omega)^k} \quad (8.9)$$

where  $\lambda_{i,k} \in \sigma(\mathbf{G}_k) \setminus \{0\}$  as before,

and  $\omega \rightarrow \infty$ .

For example if the first Markov Parameter CB has full rank  $m$  then it can be shown that the state space is the direct sum of the image of the input map  $B$  and the kernel of the output map  $C$ :

$$= \text{Im } B \oplus \text{Ker } C \quad (8.10)$$

and that the zero pencil then becomes

$$(s\mathbf{I}_{n-m} - \text{NAM})$$

There are then  $(n - m)$  finite zeros, given by  $\sigma(\text{NAM})$ , and there are  $m$  first-order asymptotic branches of the generalised Root-Locus and Nyquist diagrams given by:

$$s_{i,1} \approx -\sigma_i(\text{CB})/g \quad g \rightarrow 0$$

$$\text{and } g_{i,1} \approx - \frac{\sigma_i(\text{CB})}{j\omega} \quad i = 1, 2, \dots, m \quad \omega \rightarrow \infty. \quad (8.11)$$

where  $\{\sigma_i(\text{CB}) : i = 1, 2, \dots, m\}$  are the eigenvalues of the first Markov parameter CB.

The numbers of the various types of asymptotic behaviour are summarized in the following table (where  $v$  is the first integer for which the nullity  $d_v$  vanishes):

Order	Number of Nyquist Asymptotes	Number of Root Locus Asymptotes
1	$m - d_1$	$m - d_1$
2	$d_1 - d_2$	$2(d_1 - d_2)$
3	$d_2 - d_3$	$3(d_2 - d_3)$
$\vdots$		
$v$	$d_{v-1}$	$vd_{v-1}$
Total Number of Asymptotes	$m$	$v-1$ $m + \sum_{i=1}^{v-1} d_i$

Adding up the total number of root-locus asymptotes and subtracting the result from the dynamical order of the feedback loop we find that

$$\text{Number of finite zeros} = n - m - \sum_{i=1}^{v-1} d_i \quad (8.12)$$

A careful study of the implications of the above relationships gives a vivid geometrical insight into the role played by the basic state-space model operators  $A, B$ , and  $C$  in generating the main structural features of the asymptotic behaviour of generalised Root-Locus and Nyquist diagrams.

## 9. ANGLES OF ARRIVAL AT ZEROS AND ANGLES OF DEPARTURE FROM POLES.

The ways in which branches of the root locus depart from poles and arrive at zeros, together with the relationships of other important structural features of these diagrams to the basic state-space model parameters has been considered by a number of investigators [9], [11], [41], [42]. In this section we follow the approach of Thompson, Stein and Laub [42]. For the output-feedback system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\frac{1}{g} Ky\end{aligned}\quad (9.1)$$

the closed-loop frequency matrix will be

$$S_K(g) = A - \frac{1}{g} BKC \quad (9.2)$$

Let  $s_i, x_i$  and  $y_i^*$  be closed-loop characteristic frequencies with their associated right and left eigenvectors so that for  $0 < g < \infty$

$$\begin{aligned}(S_K(g) - s_i I) x_i &= 0 \quad i = 1, 2, \dots, n \\ y_i^* [S_K(g) - s_i I] &= 0\end{aligned}\quad (9.3)$$

then  $s_i, x_i$ , and  $y_i^*$  can be obtained by solving the generalised eigenvalue problems specified by

$$\begin{bmatrix} A - s_i I & B \\ -C & -gK^{-1} \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix} = 0 \quad i = 1, 2, \dots, p \quad (9.4)$$

$$(y_i^* \quad n_i^*) \begin{bmatrix} A - s_i I & B \\ -C & -gK^{-1} \end{bmatrix} = 0 \quad (9.5)$$

where role of the integer  $p$  is discussed below. Then, in terms of these quantities it has been shown by Thompson et al [42] that :

The angles of the root locus for  $0 < g < \infty$  and for distinct frequencies  $s_i$  are given by

$$\arg(ds_i) = \arg \left[ -\frac{y_i^* B K C x_i}{y_i^* x_i} \right] \quad 0 < g < \infty \quad i = 1, 2, \dots, p \quad (9.6)$$

or

$$\arg(ds_i) = \arg \left[ \frac{n_i^* K^{-1} w_i}{y_i^* x_i} \right] \quad 0 \leq g < \infty \quad i = 1, 2, \dots, p. \quad (9.7)$$

The angles of departure from poles are found using (9.6) with  $g = \infty$  and the angles of arrival at zeros are found using (9.7) with  $g = 0$ . Note that for  $g > 0$  we will have  $p = n$  and for  $g = 0$  we will have  $0 \leq p \leq n - m$ , where  $p$  is the number of finite zeros and  $m$  is the number of inputs and outputs of the vector feedback loop.

## 10. PROPERTIES OF NYQUIST AND ROOT LOCUS DIAGRAMS FOR OPTIMAL FEEDBACK SYSTEMS.

Several investigators have studied the asymptotic Nyquist diagram behaviour (for high frequencies), the asymptotic Root-Locus diagram behaviour (as explained below) and the location of the finite zeros for the standard optimal state feedback control problem involved in minimizing the cost function

$$J = \int_0^\infty (x^t Q x + \rho u^t R u) dt \quad (10.1)$$

where  $R$  is positive definite symmetric,  $Q$  is positive semi-definite symmetric and  $\rho$  a finite positive real constant [12], [42], [43], [44], [45], [46], [47].

It is known as a result of these studies that :

- (i) All the finite zeros of the optimal feedback loop lie in the left-half complex plane.
- (ii) All the generalised Nyquist diagrams for the optimal feedback loop have infinite gain margin and at least  $60^\circ$  phase margin.



(iii) When  $\rho$  in the performance index (10.1) is the  $n$  branches of the optimal root locus (i.e. the locus of closed-loop characteristic frequencies traced out with variation of  $\rho$ ) start on a set of poles which are the stable poles of the original plant together with the mirror images in the imaginary axis of the unstable poles of the original plant (assumed to have no purely imaginary poles). As  $\rho$  varies all the branches of the optimal root locus remain in the left-half of the frequency plane. For  $\rho$  tending to zero a number  $p$  of the branches (where  $0 \leq p \leq n-m$ ) stay finite and approach a set of loop transmission zeros. The remaining  $(m-p)$  branches approach infinity in a set of so-called Butterworth patterns. A  $k^{\text{th}}$  order Butterworth pattern has  $k$  asymptotes each of which radiates from the origin through the left-half plane solutions of

$$s^{2k} = (-1)^{k+1} \quad (10.2)$$

## 11. DESIGN TECHNIQUES.

Despite the great efforts which have been expended on the problems involved over the past decade, much remains to be done in formulating a definitive design technique for linear multivariable feedback systems of high (say having up to 100 state variables) dynamic order. A fairly comprehensive attempt to extend the philosophy and techniques of the classical Nyquist-Bode-Evans design approach to the multivariable case is given in [2], and this line of attack can be compared with a variety of other approaches to a common design problem in [48]. A fairly complete review of the development of the classical frequency-response methods can be found in [1], together with a discussion of various forms of extension to the multivariable case. A computer-aided interactive-design package [49] has been developed for use in the implementation of the techniques described in [2]. Experience of its use has shown that an interactive design method of this sort can be useful to industrial designers. There is thus a considerable incentive to develop further those forms of design technique which are based on a combination of the attractive graphical features of complex-variable methods and the geometrical way in which the main structural features of the algebraic curves used are related to state-space model parameters.

### 11.1 Design via generalised Nyquist and generalised Root Locus approach.

The configuration considered is shown in Fig.15. A given plant has  $l$  inputs and  $m$  outputs, and it is assumed that  $m > l$ . The plant output vector  $y$  consists of two subvectors  $c$  and  $z$ :

$$y = \begin{bmatrix} c \\ z \end{bmatrix}$$

where  $c$  is an  $l$ -dimensional vector of variables whose behaviour is to be controlled and  $z$  is an  $(m-l)$ -dimensional vector of extra available measurements. The design process is carried out in two main stages, corresponding to the arrangement shown in Fig.16.

**Stage 1 - Compensator Design:** The purpose of the compensator is to make use of the extra information contained in the vector  $z$  to create a new  $l$ -dimensional input vector  $v$  which gives a suitable form of transmittance from  $v$  to the set of controlled outputs  $c$ . A suitable form of this transmittance is taken to be one which allows acceptably large amounts of feedback gain to be applied around this modified plant.

**Stage 2 - Gain Injector Design:** Having used all of the available measurements to the best advantage in arranging for a suitable transmission characteristic between  $v$  and  $c$ , the design is completed by choosing a gain injector block which will inject the suitably high feedback gains required to meet the closed-loop performance specification over the operating bandwidth. The gain injector block will normally contain both integral action terms to remove low-frequency errors and phase-compensating terms to improve closed-loop damping. Early treatments of this part of the approach are often called the characteristic locus method because it is essentially based on the manipulation of characteristic gain loci (generalised Nyquist diagrams).

The basic idea behind the compensator design is best seen from a consideration of the arrangement shown in Fig.17(a) where the gain-injector block has been replaced by a unit transmittance operator. In this arrangement the compensator is designed in such a way that a suitable set of closed-loop characteristic frequencies is obtained. How the suitability of such a set is judged is suggested from a study of the implications of Fig.17. The set of closed loop characteristic frequencies associated with Fig.17(a) depends on the nature of the characteristic gain loci for the transmittance between  $v$  and  $c$  in Fig.17(b). Hence, the manipulation of the closed-loop frequencies for the arrangement of Fig.17(a) is equivalent to the manipulation of the characteristic gain loci for the compensated plant of Fig.18.

**Design of Gain Injector:** Assume that the first stage of the design procedure has been carried out; one is then considering the second-stage situation shown in Fig.19 where  $K(s)$  is the transfer-function matrix of the gain injector and  $G_c(s)$  is the transfer-function matrix of the compensated plant.

Let

$$Q(s) = G_c(s)K(s) \quad (11.1)$$

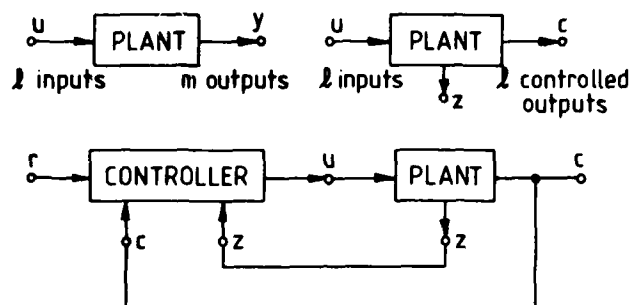


Figure 15

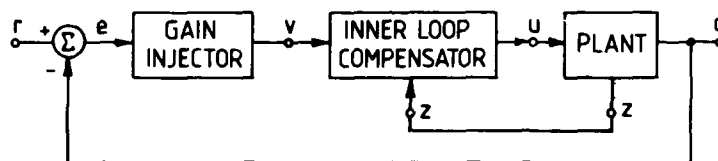
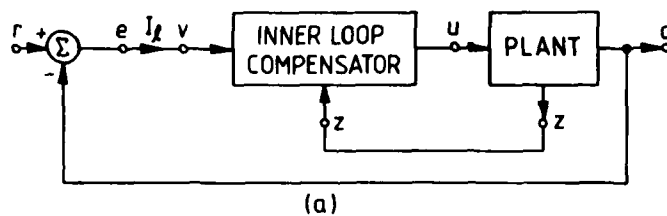
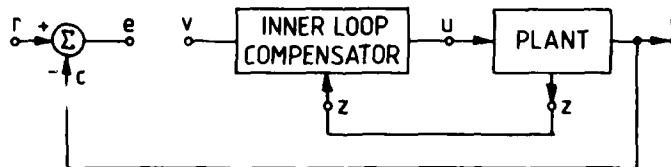


Figure 16



(a)



(b)

Figure 17

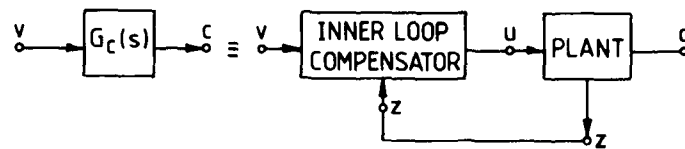


Figure 18

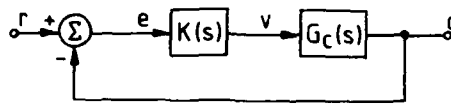


Figure 19

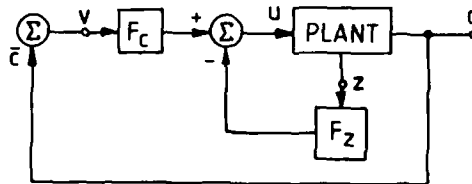


Figure 20

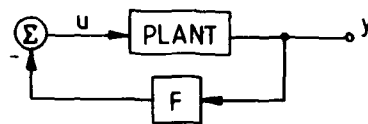


Figure 21

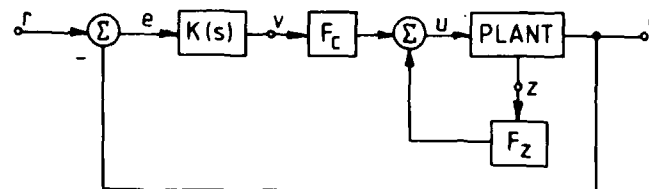


Figure 22

so that the closed-loop transfer-function matrix for this arrangement is

$$R(s) = [I_m + Q(s)]^{-1} Q(s). \quad (11.2)$$

It follows from standard relationships in matrix algebra that every eigenvector  $w_i(s)$  of  $Q(s)$  is also an eigenvector of  $R(s)$ , and that for every eigenvalue  $q_i(s)$  of  $Q(s)$  there is a corresponding eigenvalue  $q_i(s)/[1 + q_i(s)]$  of  $R(s)$ . Since the set of complex numbers  $\{q_i/(1 + q_i) : i = 1, 2, \dots, m\}$  will have two or more members with the same value if and only if the set of complex numbers  $\{q_i : i = 1, 2, \dots, m\}$  has two or more members with the same value, it follows that the set of branch points associated with the matrices  $R(s)$  and  $Q(s)$  are identical. The characteristic gain algebraic functions associated with the open-loop matrix  $Q(s)$  and the closed-loop matrix  $R(s)$  thus share the same Riemann surface, and these matrices also share the same set of eigenvectors. Therefore, if the open-loop matrix  $Q(s)$  has a dyadic expansion of the form

$$Q(s) = \sum_{i=1}^m q_i(s) w_i(s) v_i^t(s) \quad (11.3)$$

valid for almost all values of  $s$  (that is, except at the branch points), then  $R(s)$  will have a corresponding dyadic expansion of the form

$$R(s) = \sum_{i=1}^m \left[ \frac{q_i(s)}{1 + q_i(s)} \right] w_i(s) v_i^t(s) \quad (11.4)$$

valid in the same punctured plane as  $Q(s)$  (that is, the complex plane with the set of branch points deleted).

Equation (11.4) can be made the basis of a design technique for systems having the same number of inputs and outputs. The closed-loop frequency response is given by

$$R(j\omega) = \sum_{i=1}^m \left[ \frac{q_i(j\omega)}{1 + q_i(j\omega)} \right] w_i(j\omega) v_i^t(j\omega) \quad (11.5)$$

If

$$q_i(j\omega) \rightarrow \infty, \quad i = 1, 2, \dots, m,$$

then

$$R(j\omega) \rightarrow \sum_{i=1}^m w_i(j\omega) v_i^t(j\omega) = I_m,$$

where  $I_m$  is a unit matrix of order  $m$ , so that good closed-loop performance may be achieved by making all the characteristic gains of  $Q(s)$  sufficiently high over a required operating frequency range. The amounts of characteristic gain required for a given quality of closed-loop performance, and the amount of closed-loop interaction which will result, depend on the behaviour with frequency of the eigenvectors of  $Q(s)$ . The desired properties of the characteristic gain loci and characteristic gain directions (eigenvectors) of  $G_C(s)K(s)$  are therefore determined by the following considerations.

1) The characteristic gain loci (generalised Nyquist diagrams) of  $G_C(s)K(s)$  must satisfy the generalised Nyquist stability criterion.

2) For high tracking performance, and therefore low interaction, at any stipulated frequency, the gains of all the characteristic loci must be suitably large. This is particularly relevant to the low-frequency situation in which it is feasible to have high characteristic gains.

3) At high frequencies, however, it is not generally feasible, because of the excessive power requirements involved, to have large gains in the characteristic loci. Such high gains at high frequencies would, in many cases, also tend to violate the generalised Nyquist stability criterion. To reduce interaction at high frequencies, therefore, one cannot in general deploy high gains for this purpose and so must instead make the characteristic direction set of the open-loop transference align with the standard basis vector set at high frequencies.

The principal components of a design approach exploiting these ideas are, therefore,

- a) a method of manipulating characteristic gain loci, and
- b) a method of manipulating characteristic gain directions.

**Manipulation of Characteristic Gains and Directions:** From the spectral analysis point of view the essential difficulty in choosing an appropriate form of controller matrix  $K(s)$  lies in the fact that very little is known of the way in which the eigenvalues and eigenvectors of the product of two matrices  $G_C(s)$  and  $K(s)$  are related to the individual eigenvalues and eigenvectors of  $G_C(s)$  and  $K(s)$ . There is one exception to this situation, however, and this is that in which the matrices concerned commute, that is, when

$$G_C(s)K(s) = K(s)G_C(s).$$

This happens if and only if  $G_c(s)$  and  $K(s)$  have a common set of eigenvectors. In such a situation the eigenvalues of the product  $G_c(s)K(s)$  are simply appropriate products of the individual eigenvalues of the matrices  $G_c(s)$  and  $K(s)$ ; the eigenvalues which are multiplied together are those associated with a common eigenvector.

Since the equation

$$\det [gI_m - G_c(s)] = 0$$

will normally be irreducible, the eigenvalues and eigenvectors of the matrix  $G_c(s)$  will not be expressible in terms of rational functions. It will therefore not normally be feasible to construct a realizable controller  $K(s)$  which will exactly commute with  $G_c(s)$  for all values of  $s$ . Even in the rare cases when this might be possible, such a controller could be unnecessarily complicated. A more practical and rewarding approach is to investigate the possibility of using controllers which are approximately commutative with the plant at appropriately chosen values of frequency. In one approach of this sort (MacFarlane and Kouvaritakis [50]) the controller matrix is chosen to have the specific form

$$K(s) = H\Lambda_k(s)J$$

where  $H$  and  $J$  are matrices with real elements and  $\Lambda_k(s)$  is a diagonal matrix of rational functions in  $s$ . The matrix  $K(s)$  can therefore be expressed in the dyadic form

$$K(s) = \sum_{j=1}^m k_j(s) h_j j_i^t$$

where the vector sets

$$\{h_j : j = 1, 2, \dots, m\} \text{ and } \{j_i^t : i = 1, 2, \dots, m\}$$

are both real. An approximately-commutative controller is one whose constituent real matrices  $H$  and  $J$  are chosen in such a way that  $K(s)$  is approximately commutative with  $G_c(s)$  at some specific value of complex frequency  $\bar{s}$ . At such a frequency one has that

$$q_i(\bar{s}) = g_i(\bar{s})k_i(\bar{s}), \quad i = 1, 2, \dots, m$$

where  $q_i(\bar{s}), g_i(\bar{s})$  are eigenvalues of  $Q(\bar{s})$  and  $G_c(\bar{s})$ , respectively, and  $k_i(\bar{s})$  are appropriate diagonal elements of  $\Lambda_k(\bar{s})$ . This set of relationships gives an obvious basis for manipulating the characteristic gain loci of  $Q(s)$  in a systematic manner; it also turns out that this approximately commutative technique can be used to manipulate the eigenvectors of  $Q(s)$ . Such manipulation techniques have been described by MacFarlane and Kouvaritakis [50].

**Design of the Inner-Loop Compensator:** The ideas behind the design of the inner-loop compensator are illustrated by Figs. 20-22. The form of the characteristic gain loci for the compensated plant  $G_c(s)$ , that is, for the transmittance between  $v$  and  $c$  in Fig. 17(b), is directly related to the nature of the closed-loop characteristic frequencies for the arrangement of Fig. 17(a), which in turn is equivalent to Figs. 20 and 21. Thus the problem of choosing the component blocks  $F_c$  and  $F_o$  of Fig. 20 for the compensator is essentially the problem of choosing the feedback matrix  $F$  in the configuration of Fig. 21 in such a way that a suitable set of closed-loop characteristic frequencies is obtained. Since  $F$  has  $m$  inputs and  $l$  outputs, where  $l < m$ , the choice of  $F$  is often referred to as a "squaring-down procedure". The choice of  $F$  is essentially based on generalised root-locus ideas and involves the placement of finite zeros and the manipulation of root-locus asymptotes (MacFarlane, Kouvaritakis, and Edmunds [10]). The final objective of the "inner-loop compensator" design is to get suitable locations for the closed-loop poles seen by the gain injector  $K(s)$  in Fig. 22. A satisfactory achievement of this objective requires some dynamical insight and engineering judgment since, in the usual case when all the plant states are not directly accessible, it will not normally be possible to place finite zeros in arbitrarily specified locations in the complex plane. Nevertheless, the basic ideas underlying the technique of inner-loop compensator design are quite straightforward: one places finite zeros and manipulates root-locus asymptote patterns in such a way that, by setting a gain parameter at a suitable value, the closed-loop poles of the system "seen by" the gain-injector  $K(s)$  are pulled into suitable locations in the complex plane. In some circumstances one may wish to pull such closed-loop poles well over into the left-half complex plane in order to speed up closed-loop response and to allow  $K(s)$  to inject a desired amount of gain before the overall (inner plus outer loops) system has its closed-loop characteristic frequencies driven into unacceptable regions of the complex plane. In other circumstances one may wish to "freeze" the location of major poles associated with certain open-loop plant characteristic frequencies by placing finite zeros near them. Often a judicious mixture of both approaches will be required.

Although it requires engineering insight and manipulative skill, it has been shown to be a powerful tool in the hands of a skilled designer. A fairly extensive computer-aided design package has been developed for use in this way [49], [51].

## 11.2 Least-squares Nyquist/Bode Array Approach.

An alternative approach has been developed by Edmunds [52] which transfers the burden of the designer's choice to the selection of a closed-loop performance specification and a controller structure. This approach uses Nyquist and Bode diagram arrays of the open-loop and closed-loop behaviour as a means of assessing performance and interaction. Following a preliminary analysis of the problem (using the techniques of Section 11.1 above) the designer chooses a suitable specification for the closed-loop behaviour of the feedback system together with a stipulated structure for the feedback controller to be used. The controller structure selected has a number of variable parameters and the design is completed by a computer optimization (using say a standard weighted least-squares algorithm) of the difference between the specified and actual closed-loop performances. As one would expect this method, when properly used, produces better results than that of Section 11.1 above. It has the further advantage of being extendable to deal with variations in the system model parameters. The CAD package mentioned above [49], [51] incorporates the routines for dealing with this approach.

## 11.3 Example of frequency-response design approach.

Fig.23 shows the Nyquist array for a 2-input, 2 out-put, 12-state automotive gas turbine. Since all the elements of this array are of the same order of magnitude, output scaling as an initial step in the design procedure is not required. The characteristic loci (generalised Nyquist diagrams) for this system are shown in Fig.24. An inspection of these immediately shows that no serious design difficulties should arise; the speed with which such an initial assessment can be carried out is one of the great advantages of the generalised frequency-response approach. The next step in the procedure for this example was to carry out a 'high frequency alignment' [2] at a frequency of 3 rad/sec. The frequency used is usually that corresponding to the target closed-loop bandwidth. Figs. 25 and 26 show the generalised Nyquist and Bode plots of the system's characteristic gain functions after this alignment step. An inspection of these Bode plots shows that one of the characteristic gains is fairly 'flat' near the critical frequency, so some extra phase compensation  $(s+5)/(s+1)$  is next added to this locus using an 'approximately commutative controller' [2] calculated at a frequency of 3 rad/sec. Fig. 27 shows the 'misalignment angles' [2] following the high-frequency alignment procedure. It can be seen that the misalignment angles have been suitably reduced except for frequencies in the region of 1.5 to 2 rad/sec.

Fig.28 shows Bode plots of the characteristic gains with an alignment at 3 rad/sec. followed by the approximately commutative controller at 3 rad/sec., followed by a further alignment at 3 rad/sec. It can be seen that the gains are now reasonably well balanced. To complete the design extra compensation is now added for low and medium frequencies. A proportional-plus-integral control term  $(s+1)/s$  is chosen for the lower-gain locus at lower frequencies. The higher-gain locus needs some extra phase advance at frequencies just below 1 rad/sec. and hence a term of  $[(s+0.2)(s+0.4)]/[s(s+1.5)]$  is used for it. This compensation is injected via an 'approximately commutative controller' at a frequency of 0.7 rad/sec.

The generalised Nyquist and Bode diagrams for the final compensated system (with an align at 3, a commutative controller at 3, a further align at 3, and a final commutative controller at 0.7) are shown in Figs. 29 and 30. These show good stability margins and gains which are well-balanced over a wide range of frequencies. Fig.31 shows the closed-loop gain array (in Bode form); from this we see that the closed-loop system has a bandwidth of about 10 rad/sec. in the first loop and of about 3 rad/sec. in the second loop. Fig. 32 shows the magnitudes of the principal gains of the closed-loop system as a function of frequency. These show that the system will be reasonably robust against multiplicative perturbations in the open-loop transfer-function.

## ACKNOWLEDGEMENTS

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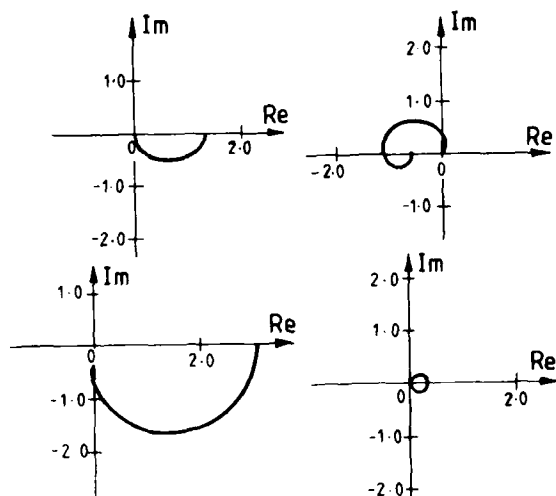


Figure 23

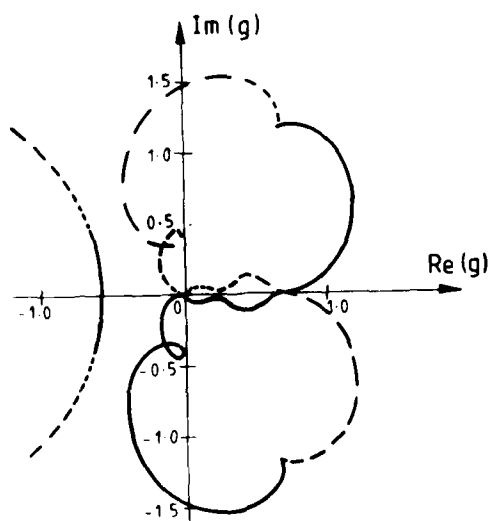


Figure 24

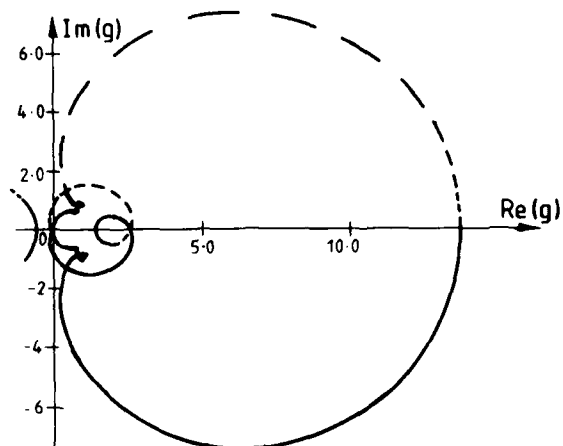


Figure 25

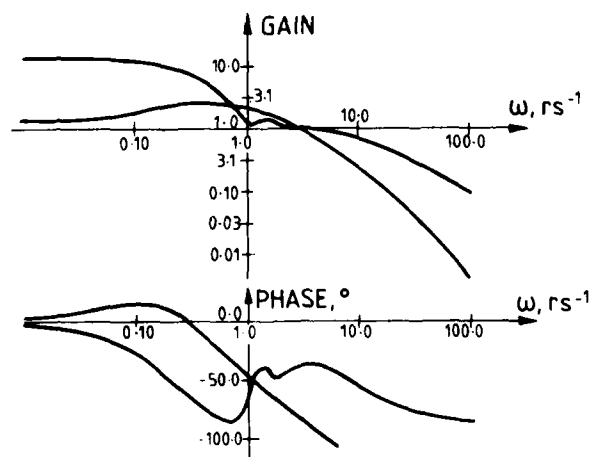


Figure 26

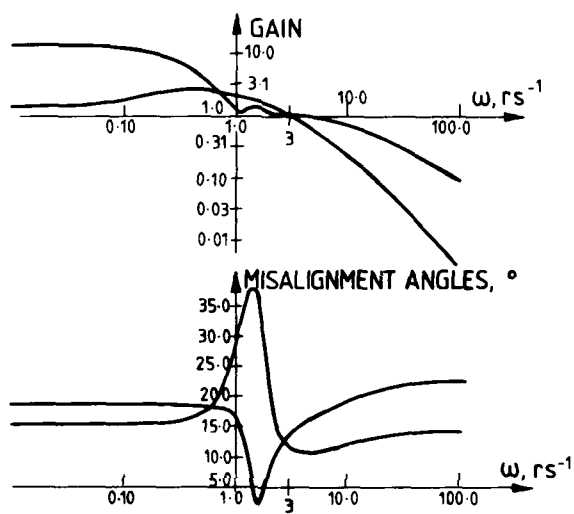


Figure 27

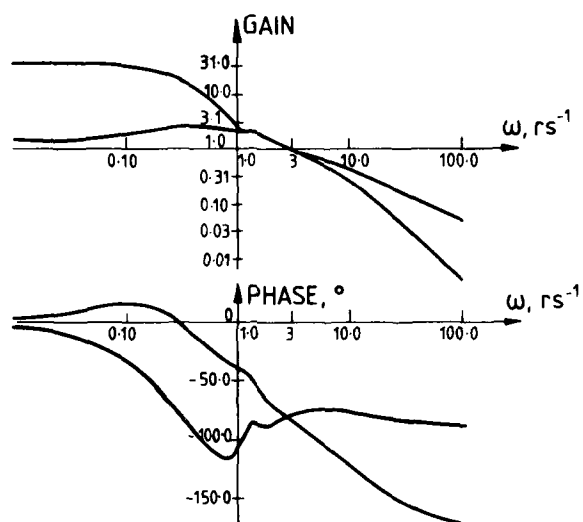


Figure 28

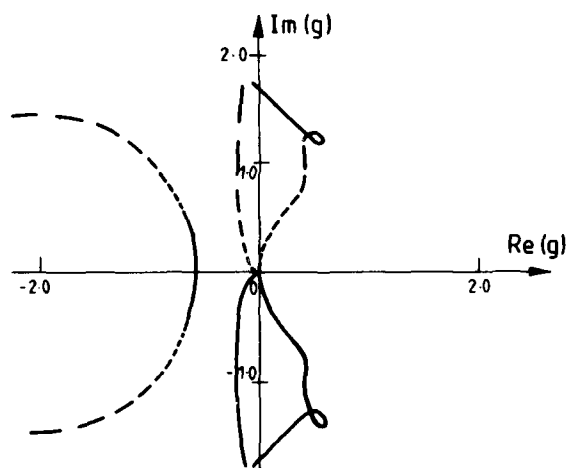


Figure 29

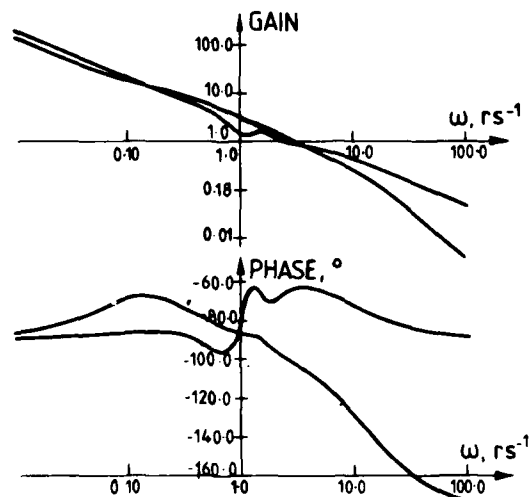


Figure 30

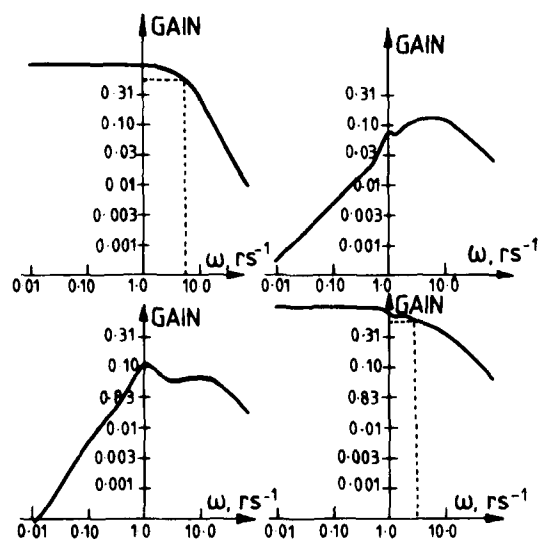


Figure 31

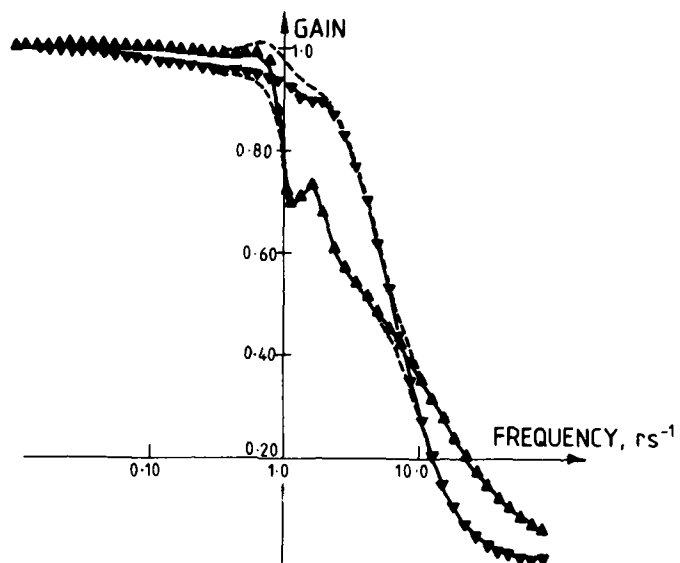


Figure 32

# Multivariable Design Techniques Based on Singular Value Generalizations of Classical Control

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## SUMMARY

This paper uses singular value analysis to generalize the fundamental feedback ideas from classical single loop control theory to multiloop systems. The classical view of measuring the benefit of feedback in terms of desensitizing the system to plant variations and disturbances is discussed. Uncertainty is shown to play a critical role in determining the way in which feedback may be used. Certain singular value plots, called  $\sigma$ -plots, are introduced as natural and effective generalizations of Bode gain plots and form the basic tools for analysis of multiloop feedback systems. These tools provide reliable means for assessing the stability margins, bandwidth, and desensitizing effects of multiloop feedback systems. Examples are given to illustrate the use of the  $\sigma$ -plots and their interpretations. Most of this paper is excerpted from other papers already published ([1] - [4]).

## I. Introduction

In designing control systems, there are fundamental issues that transcend the boundaries of specific applications. Although they may be packaged slightly differently for each application and may have different levels of importance, these issues are generic in their relationship to control design objectives and procedures. Central to these issues is the control requirement to provide satisfactory performance in the face of system variations and uncertainty. This requirement was the original motivation for the development of feedback systems. Feedback is only required when system performance cannot be achieved because of uncertainty in system characteristics. This paper is based on the premise that the recognition and proper treatment of uncertainty holds the key to viable solutions for essentially all feedback design problems. Section II qualitatively discusses the relationship between design models and the real world. This includes how the use of models, with their associated uncertainties, impact the control design process.

Section III reviews the fundamental practical issue in feedback design -- namely, how to achieve the benefits of feedback in the face of uncertainties. The nominal design models used are assumed to be finite dimensional, linear and time variant (FDLTI) and the approach is basically classical ([5] - [6]) expressed in terms of transfer functions in the frequency domain. Singular values are used to generalize the classical single-input single-output (SISO) results to multi-input multi-output (MIMO) systems.

In Section IV the problem of representing uncertainty in the design model is discussed in more detail. Various types of uncertainties which can arise in physical systems are briefly described and "unstructured uncertainties" are singled out as generic errors which are associated with all design models. Section V then shows how classical SISO statements of the feedback design problem in the face of unstructured uncertainty can be reliably generalized to MIMO system. Section VI presents some examples to illustrate the concepts discussed in the preceding sections. Conclusions follow in the last Section VIII.

## II. Modeling and Uncertainty

Most control designs are based on the use of a design model. The relationship between models and the reality they represent is subtle and complex. Thus, the problems created by model uncertainty have often been either trivialized or ignored in theoretical studies in favor of assuming the alternative of no distinction between models and reality.

A mathematical model provides a map from inputs to responses. The quality of a model depends on how closely its responses match those of the true plant. Since no single fixed model can respond exactly like the true plant, we need, at the very least, a set of maps. However, the modeling problem is much deeper -- the universe of mathematical models from which a model set is chosen is distinct from the universe of physical systems. Therefore, a model set can never be constructed which includes the true physical plant. It is necessary for the engineer to make a leap of faith regarding the applicability of a particular design based on a mathematical model. To be practical, a



design technique must help make this leap small by accounting for the inevitable inadequacy of models. A good model should be simple so as to facilitate design, yet complex enough to give the engineer confidence that designs based on the model will work on the true plant.

The term uncertainty refers to the differences or errors between models and reality and whatever mechanism is used to express these errors will be called a representation of uncertainty. For example, consider the problem of bounding the magnitude of the effect of some uncertainty on the output of a nominally fixed linear system. A useful measure of uncertainty in this context is to provide a bound on the spectrum of the deviation of the output from its nominal response. In the simplest case, this spectrum is assumed to be independent of the input. This is equivalent to assuming that the uncertainty is generated by an additive noise signal with bounded spectrum; the uncertainty is represented as additive noise. Of course, no physical system is linear with additive noise, but some aspects of physical behavior are approximated quite well using this model. This type of uncertainty has received a great deal of attention in the literature, perhaps more because it yields elegant theoretical solutions (e.g., white noise propagation in linear systems, Wiener and Kalman filtering, LQG) than because it is of greater practical significance than other types of uncertainty.

Generally, the spectrum of the deviation of the true output from the nominal will depend significantly on the input. For example, an additive noise model is entirely inappropriate for capturing uncertainty arising from variations in the material properties of physical plants. The actual construction of model sets for more general uncertainty can be quite difficult. Suppose we begin with a differential equation for our model. We immediately recognize the need to parametrize the differential equation to reflect plant variations relative to our model. However, for certain classes of signals (e.g., high frequency) the parametrized differential equation fails to describe the plant, because the plant will always have dynamics which are not represented in the differential equation.

In general, we are forced to use model sets that allow for plant dynamics which are not explicitly represented in the model structure. A simple example of this involves using frequency-domain bounds on transfer functions to describe a model set. To use such sets to describe physical systems, the bounds must roughly grow with frequency. In particular, at sufficiently high frequencies, phase is completely unknown, i.e.,  $\pm 180^\circ$  uncertainties. This is a consequence of dynamic properties which inevitably occur in physical systems. These issues will be discussed further in Section IV.

Assuming we are given a model including a representation of uncertainty, which we believe adequately captures the essential features of the plant, the next step in the controller design problem is to determine what structure is necessary to achieve the desired performance. Prefiltering of input signals can change the dynamic response of the model set but cannot reduce the effect of uncertainty. If the uncertainty is too great to achieve the desired accuracy of response, then a feedback structure is required. The mere assumption of a feedback structure, however, does not guarantee a reduction of uncertainty, and there are many obstacles to achieving the uncertainty-reducing benefits of feedback.

In particular, since for any reasonable model set representing a physical system, uncertainty becomes large and phase is completely unknown at sufficiently high frequencies, the loop gain must be small at those frequencies to avoid destabilizing the high frequency system dynamics. Even worse is that the feedback system actually increases uncertainty and sensitivity in the frequency ranges where uncertainty is sufficiently large. In other words, because of the type of sets required to reasonably model physical systems and the restriction that our controllers be causal, we cannot use feedback (or any other control structure) to cause our closed-loop model set to be a proper subset of the open-loop model set. Often, what can be achieved with intelligent use of feedback is a significant reduction of uncertainty for certain signals of importance with a small increase spread over other signals. Thus, the feedback design problem centers around the tradeoff involved in reducing the overall impact of uncertainty. This tradeoff also occurs for example, when, using feedback to reduce command/disturbance error while minimizing response degradation due to measurement noise. To be of practical value, a design technique must provide means for performing these tradeoffs.

### III. The Benefits of Feedback

We will deal with the standard feedback configuration illustrated in Figure 1:

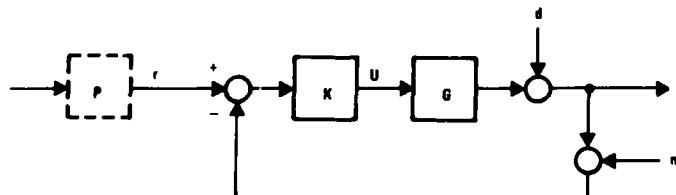


Figure 1: Standard Feedback Configuration

It consists of the interconnected plant (G) and controller (K) forced by commands (r), measurement noise (n), and disturbances (d). The dashed precompensator (P) is an optional element used to achieve deliberate command shaping or to represent a non-unity feedback system in equivalent unity feedback form. All disturbances are assumed to be reflected to the measured output (y), all signals are multivariable, in general, and both nominal mathematical models for G and K are finite dimensional linear time invariant (FDLTI) systems with transfer functions matrices G(s) and K(s). Then it is well known that the configuration, if it is stable, has the following major properties:

(1) Input-output behavior

$$y = GK(I + GK)^{-1} (r - n) + (I + GK)^{-1}d \quad (1)$$

$$e \triangleq r - y$$

$$= (I + GK)^{-1} (r - d) + GK(I + GK)^{-1}n \quad (2)$$

(2) System sensitivity [7]

$$\Delta H_{cl} = (I + G'K)^{-1}\Delta H_{ol} \quad (3)$$

In equation (3),  $\Delta H_{cl}$  and  $\Delta H_{ol}$  denote changes in the closed loop system and changes in a nominally equivalent open loop system, respectively, caused by changes in the plant G, i.e.,  $G' = G + \Delta G$ .

Equations (1) through (3) summarize the fundamental benefits and design objectives inherent in feedback loops. Specifically, equation (2) shows that the loop's errors in the presence of commands and disturbances can be made "small" by making the sensitivity operator, or inverse return difference operator,  $(I + GK)^{-1}$ , "small", and equation (3) shows that loop sensitivity is improved under these same conditions, provided G' does not stray too far from G. An alternative approach which measures sensitivity in terms of G and not G' is given in the companion paper [19].

For SISO systems, the appropriate notion of smallness for the sensitivity operator is well understood - namely, we require that the complex scalar  $[1 + g(j\omega) k(j\omega)]^{-1}$  have small magnitude, or equivalently that  $1 + g(j\omega) k(j\omega)$  have large magnitude, for all real frequencies  $\omega$  where the commands, disturbances and/or plant changes,  $\Delta G$ , are significant. In fact, the performance objectives of SISO feedback systems are commonly stipulated in terms of explicit inequalities of the form

$$ps(\omega) \leq |1 + g(j\omega) k(j\omega)| \quad \forall \omega < \omega_0, \quad (4)$$

where  $ps(\omega)$  is a (large) positive function and  $\omega_0$  specified the active frequency range.

This basic idea can be readily extended to MIMO problems through the use of matrix norms. Selecting the spectral norm as our measure of matrix size, for example, the corresponding feedback requirements become

$$\overline{\sigma} [(I + G(j\omega) K(j\omega))^{-1}] \text{ small}$$

or conversely

$$ps(\omega) \leq \underline{\sigma} [I + G(j\omega) K(j\omega)] \quad (5)$$

for the necessary range of frequencies. The symbols  $\overline{\sigma}$  and  $\underline{\sigma}$  in these expressions are defined as follows:

$$\overline{\sigma} [A] \triangleq \max_{\|x\|=1} \|Ax\| = \sqrt{\lambda_{\max}[A^*A]} \quad (6)$$

$$\underline{\sigma} [A] \triangleq \min_{\|x\|=1} \|Ax\| = \sqrt{\lambda_{\min}[A^*A]} \quad (7)$$

where  $\| \cdot \|$  is the usual Euclidian norm,  $\lambda[\cdot]$  denotes eigenvalues, and  $[\cdot]^*$  denotes conjugate transpose. The two  $\sigma$ 's are called maximum and minimum singular values of A (or principal gains [4]), respectively, and can be calculated with available linear system software.[8] More discussion of singular values and their properties can be found in various texts.[9]

Condition (5) on the return difference  $I + GK$  can be interpreted as merely a restatement of the common intuition that large loop gains or "tight" loops yield good performance. This follows from the inequalities

$$\underline{\sigma}[GK] - 1 \leq \underline{\sigma}[I+GK] \leq \underline{\sigma}[GK]+1 \quad (8)$$

which show that return difference magnitudes approximate the loop gains,  $\underline{\sigma}[GK]$ , whenever these are large compared with unity. Evidently, good multivariable feedback loop design boils down to achieving high loop gains in the necessary frequency range.

Despite the simplicity of this last statement, it is clear from years of research and design activity that feedback design is not trivial. This is true because loop gains cannot be made arbitrarily high over arbitrarily large frequency ranges. Rather, they

must satisfy certain performance tradeoffs and design limitations. A major performance tradeoff, for example, concerns command and disturbance error reduction versus sensor noise error reduction. [10] The conflict between these two objectives is evident in equation (2). Large  $\sigma[G(j\omega)]$  values over a large frequency range make errors due to  $r$  and  $d$  small. However, they also make errors due to  $n$  large because this noise is "passed through" over the same frequency range, i.e.,

$$y = GK(j\omega) \{I + GK(j\omega)\}^{-1} n \approx In \quad (9)$$

Worse still, large loop gains can make the control activity (variable  $u$  in Figure 1) quite unacceptable. This follows from

$$u = K[I + GK]^{-1}(r - n - d) \approx G^{-1}(j\omega)(r - n - d) \quad (10)$$

Here we have assumed  $G$  to be square and invertible for convenience. The resulting equation shows that commands, disturbances and sensor noise are actually amplified at  $u$  whenever the frequency range significantly exceeds the bandwidth of  $G$ ; i.e., for  $\omega$  such that  $\sigma[G(j\omega)] \ll 1$  we get

$$\sigma[G^{-1}(j\omega)] = \frac{1}{\sigma[G(j\omega)]} \gg 1 \quad (11)$$

One of the major contributions of modern feedback theory is the development of systematic procedures for conducting the above performance tradeoffs. We are referring, of course, to the LQG theory [11] and to its modern Wiener-Hopf frequency domain counterpart. [12] Under reasonable assumptions on plant, disturbances, and performance criteria, these procedures yield efficient design compromises. In fact, if the tradeoff between command/disturbance error reduction and sensor noise error reduction were the only constraint on feedback design, practitioners would have little to complain about with respect to the relevance of modern theory. The problem is that these performance tradeoffs are often overshadowed by a second limitation on high loop gains -- namely, the requirement for tolerance to uncertainties. Though a controller may be designed using FDLTI models, the design must be implemented and operate with a real physical plant. The properties of physical systems, in particular the ways in which they deviate from finite-dimensional linear models, put strict limitations on the frequency range over which the loop gains may be large. In order to properly motivate these restrictions, we digress in Section 3 to a brief description of the types of system uncertainties most frequently encountered. The manner in which these uncertainties can be accounted for in MIMO design then forms the basis for the rest of the paper.

#### IV. The Nature of Uncertainty

While no nominal design model,  $G(s)$ , can emulate a physical plant perfectly, it is clear that some models do so with greater fidelity than others. Hence, no nominal model should be considered complete without some assessment of its errors. We will call these errors the "model uncertainties", and whatever mechanism is used to express them will be called a "representation of uncertainty."

Representations of uncertainty vary primarily in terms of the amount of structure they contain. This reflects both our knowledge of the physical mechanisms which cause differences between model and plant and our ability to represent these mechanisms in a way that facilitates convenient manipulation. For example, a set membership statement for the parameters of an otherwise known FDLTI model is a highly-structured representation of uncertainty. It typically arises from the use of linear incremental models at various operating points, e.g., aerodynamic coefficients in flight control vary with flight environment and aircraft configurations, and equation coefficients in power plant control vary with aging, slag buildup, coal composition, etc. In each case, the amounts of variation and any known relationships between parameters can be expressed by confining the parameters to appropriately defined subsets of parameter space. A specific example of such a parameterization for the F-8C aircraft is given in [13].

Examples of less-structured representations of uncertainty are direct set membership statements for the transfer function matrix of the model. For instance, the statement

$$G'(j\omega) = G(j\omega) + \Delta G(j\omega)$$

with

$$\sigma[\Delta G(j\omega)] < \ell_a(\omega) \quad \forall \omega \geq 0 \quad (12)$$

where  $\ell_a(\cdot)$  is a positive scalar function, confines the matrix  $G'$  to a neighborhood of  $G$  with magnitude  $\ell_a(\omega)$ . The statement does not imply a mechanism or structure which gives rise to  $\Delta G$ . The uncertainty may be caused by parameter changes, as above, or by neglected dynamics, or by a host of other unspecified effects. An alternative statement to (12) is the so-called multiplicative form:

$$G'(j\omega) = \{I + L(j\omega)\}G(j\omega)$$

with

$$\sigma[L(j\omega)] < \ell_m(\omega) \quad \forall \omega \geq 0 \quad (13)$$

This statement confines  $G'$  to a normalized neighborhood of  $G$ . An advantage of (13) over (12) is that in (13) compensated transfer functions have the same uncertainty representation as the raw model (i.e., the bound (13) applied to  $GK$  as well as to  $G$ ). Still other alternative set membership statements are the inverse forms of (12) and (13) which confine  $(G')^{-1}$  to direct or normalized neighborhoods about  $G^{-1}$ .

The best choice of uncertainty representation for a specific FDLTI model depends, of course, on the errors the model makes. In practice, it is generally possible to represent some of these errors in a highly-structured parameterized form. These are usually the low frequency error components. There are always remaining higher frequency errors, however, which cannot be covered this way. These are caused by such effects as infinite-dimensional electro-mechanical resonances [16, 17], time delays, diffusion processes, etc. Fortunately, the less-structured representations, (12) or (13), are well suited to represent this latter class of errors. Consequently, (12)-(13) have become widely used "generic" uncertainty representations for FDLTI models.

Motivated by these observations, we will focus throughout the rest of this paper exclusively on the effects of uncertainties as represented by (13). For lack of a better name, we will refer to these uncertainties simply as "unstructured." We will assume that  $G'$  in (13) remains a strictly proper FDLTI system and that  $G'$  has the same number of unstable modes as  $G$ . The unstable modes of  $G'$  and  $G$  do not need to be identical, however, and hence  $L(s)$  may be an unstable operator. These restricted assumptions on  $G'$  make exposition easy. More general perturbations (e.g., time varying, infinite dimensional, nonlinear) can also be covered by the bounds in (13) provided they are given appropriate "conic sector" interpretations via Parseval's theorem. This connection is developed in [14, 15] and will not be pursued here.

When used to represent the various high frequency mechanisms mentioned above, the bounding functions  $\ell_m(\omega)$  in (13) commonly have the properties illustrated in Figure 2. They are small ( $\ll 1$ ) at low frequencies and increase to unity and above at higher frequencies. The growth with frequency inevitably occurs because phase uncertainties eventually exceed  $\pm 180$  degrees and magnitude deviations eventually exceed the nominal transfer function magnitudes. Readers who are skeptical about this reality are encouraged to try a few experiments with physical devices.

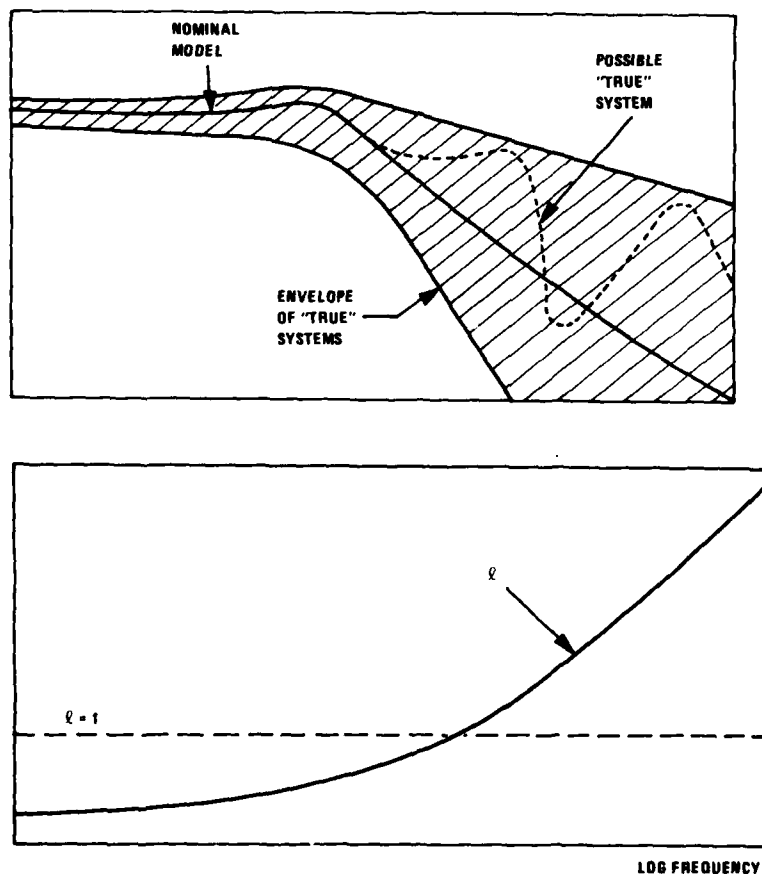


Figure 2. Typical Behavior of Multiplicative Perturbations

It should also be noted that the representation of uncertainty in (13) can be used to include perturbation effects that are in fact not at all certain. A nonlinear element, for example, may be quite accurately modeled, but because our design techniques cannot deal with the nonlinearity effectively, it is treated as a conic linearity. [14, 15] As another example, we may deliberately choose to ignore various known dynamic characteristics in order to achieve a simpler nominal design model.

Another important point is that the construction of  $\underline{L}_m(\omega)$  for multivariable systems is not trivial. The bound assumes a single worst case uncertainty magnitude applicable to all channels. If substantially different levels of uncertainty exist in various channels, it may be necessary to scale the input-output variables and/or apply frequency-dependent transformations [15] in such a way that  $\underline{L}_m$  becomes more uniformly tight. These scale factors and transformations are here assumed to be part of the nominal model  $G(s)$ .

#### V. Feedback Design in the Face of Unstructured Uncertainties

Once we specify a design model,  $G(s)$ , and accept the existence of unstructured uncertainties in the form (13), the feedback design problem becomes one of finding a compensator  $K(s)$  such that

- (i) the nominal feedback system,  $GK[I + GK]^{-1}$ , is stable;
- (ii) the perturbed system,  $G'K[I + G'K]^{-1}$ , is stable for all possible  $G'$  allowed by (13); and
- (iii) performance objectives are satisfied for all possible  $G'$  allowed by (13).

All three of these requirements can be interpreted as frequency domain conditions on the nominal loop transfer matrix,  $GK(s)$ , which the designer must attempt to satisfy.

#### Stability Conditions

The frequency domain conditions for Requirement (i) are, of course, well known. In SISO cases, they take the form of the standard Nyquist Criterion, [See any classical control text.] and in MIMO cases, they involve its multivariable generalization. [18] Namely, we require that the encirclement count of the map  $\det [I + GK(s)]$ , evaluated on the standard Nyquist D-contour, be equal to the (negative) number of unstable open loop modes of  $GK$ .

Similarly, for Requirement (ii) the number of encirclements of the map  $\det [I + G'K(s)]$  must equal the (negative) number of unstable modes of  $G'K$ . Under our assumptions on  $G'$ , however, this number is the same as that of  $GK$ . Hence, Requirement (ii) is satisfied if and only if the number of encirclements of  $\det [I + G'K(s)]$  remains unchanged for all  $G'$  allowed by (13). This is assured if  $\det [I + G'K]$  remains nonzero as  $G$  is warped continuously toward  $G'$ , or equivalently, iff

$$0 < \underline{\sigma} [I + \epsilon L(s) | G(s) K(s)] \quad (14)$$

for all  $0 \leq \epsilon \leq 1$ , all  $s$  on the D-contour, and all  $L(s)$  satisfying (13). Since  $G'$  vanishes on the infinite segment of the D-contour, and assuming, for simplicity, that the contour requires no indentations along the  $j\omega$ -axis, (If indentations are required, (14) and (17) must hold in the limit for all  $s$  on the indented path as the radius of indentation is taken to zero) equation (14) reduces to the following equivalent conditions:

$$0 < \underline{\sigma} [I + G(j\omega) K(j\omega) + \epsilon L(j\omega) G(j\omega) K(j\omega)] \quad (15)$$

for all  $0 \leq \epsilon \leq 1$ ,  $0 \leq \omega < \infty$ , and all  $L$

$$\Leftrightarrow 0 < \underline{\sigma} [I + L GK(I + GK)^{-1}] \quad (16)$$

for all  $0 \leq \omega < \infty$ , and all  $L$

$$\Leftrightarrow \bar{\sigma} [GK(I + GK)^{-1}] < 1/\underline{L}_m(\omega) \quad (17)$$

for all  $0 \leq \omega < \infty$

The last of these equations is the MIMO generalization of the familiar SISO requirement that loop gains be small whenever the magnitude of unstructured uncertainties is large. In fact, whenever  $\underline{L}_m(\omega) \gg 1$ , we get the following constraint on  $GK$ :

$$\bar{\sigma} [GK(j\omega)] < 1/\underline{L}_m(\omega) \quad (18)$$

for all  $\omega$  such that  $\underline{L}_m(\omega) \gg 1$ .

Note that these are not conservative stability conditions. On the contrary, if the uncertainties are truly unstructured and (17) is violated, then there exists a perturbation  $L(s)$  within the set allowed by (13) for which the system is unstable. Hence, these stability conditions impose hard limits on the permissible loop gains of practical feedback systems.

### Performance Conditions

Frequency domain conditions for Requirement (iii) have already been described in Section 2, equation (5). The modification needed to account for unstructured uncertainties is to apply (5) to  $G'$  instead of  $G$ ; i.e.,

$$\begin{aligned} p_s &\leq \underline{\sigma} [I + (I+L)GK] \\ &\Leftarrow p_s \leq \underline{\sigma} [I+L GK(I+GK)^{-1}] \leq \underline{\sigma} [I+GK] \\ &\Leftarrow \frac{p_s(\omega)}{1-l_m(\omega)} + 1 \leq \underline{\sigma} [GK(j\omega)] \end{aligned} \quad (19)$$

for all  $\omega$  such that  $l_m(\omega) < 1$  and  $\underline{\sigma} [GK(j\omega)] \gg 1$ .

This is the MIMO generalization of another familiar SISO design rule -- namely that performance objectives can be met in the face of unstructured uncertainties if the nominal loop gains are made sufficiently large to compensate for model variations. Note, however, that finite solutions exist only in the frequency range where  $l_m(\omega) < 1$ .

The stability and performance conditions derived above illustrate that MIMO feedback design problems do not differ fundamentally from their SISO counterparts. In both cases, stability must be achieved nominally and assured for all perturbations by satisfying conditions (17-18). Performance may then be optimized by satisfying condition (19) as well as possible. What distinguishes MIMO from SISO design conditions are the functions used to express transfer function "size." Singular values replace absolute values. The underlying concepts remain the same.

We note that the singular value functions used in our statements of design conditions play a design role much like classical Bode plots. The  $\underline{\sigma}[I + GK]$  function in (5) is the minimum return difference magnitude of the closed loop system,  $\underline{\sigma}[GK]$  in (8) and  $\underline{\sigma}[GK]$  in (18) are minimum and maximum loop gains, and  $\underline{\sigma}[GK(I + GK)^{-1}]$  in (17) is the maximum closed loop frequency response. These can all be plotted as ordinary frequency dependent functions in order to display and analyze the features of a multivariable design. Such plots will here be called  $\sigma$ -plots.

One of the  $\sigma$ -plots which is particularly significant with regard to design for uncertainties is obtained by inverting condition (17), i.e.,

$$l_m(\omega) < \frac{1}{\underline{\sigma}[GK(I+GK)^{-1}]} = \underline{\sigma} [I + (GK(j\omega))^{-1}] \quad (20)$$

for all  $0 \leq \omega < \infty$ . The function on the right hand side of this expression is an explicit measure of the degree of stability (or stability robustness) of the feedback system. Stability is guaranteed for all perturbations  $L(s)$  whose maximum singular values fall below it. This can include gain or phase changes in individual output channels, simultaneous changes in several channels, and various other kinds of perturbations. In effect,  $\underline{\sigma}[I + (GK)^{-1}]$  is a reliable multivariable generalization of SISO stability margin concepts (e.g., frequency dependent gain and phase margins). Unlike the SISO case, however, it is important to note that  $\underline{\sigma}[I + (GK)^{-1}]$  measures tolerances for uncertainties at the plant outputs only.

Tolerances for uncertainties at the input are generally not the same. They can be analyzed with equal ease, however, by using the function  $\underline{\sigma}[I + (KG)^{-1}]$  instead of  $\underline{\sigma}[I + (GK)^{-1}]$  in (20). This can be readily verified by evaluating the encirclement count of the map  $\det(I + KG)$  under perturbations of the form  $G' = G(I + L)$  (i.e., uncertainties reflected to the input). The mathematical steps are directly analogous to (15-18) above. Classical designers will recognize, of course, that the difference between these two stability robustness measures is simply that each uses a loop transfer function appropriate for the loop-breaking point at which robustness is being tested. The relationship between input and output margins is discussed further in the companion paper [19].

The feedback design conditions derived above can be related directly to the classical feedback problem as formulated by Bode ([5], [6]). This problem is pictured graphically for the SISO case in Figure 3 using the well-known Bode gain plot. Note the low and high frequency gain conditions. The designer must find a loop transfer function,  $GK$ , for which the loop is nominally stable and whose gain clears the high and low frequency "design boundaries" given by Condition (17) and (19). The high frequency boundary is mandatory, while the low frequency one is desirable for good performance. Both are influenced by the uncertainty bound,  $l_m(\omega)$ .

A representative loop transfer function is also sketched in the figure. As shown, the effective bandwidth of the loop cannot fall much beyond the frequency  $\omega_0$  for which  $l_m(\omega_0) = 1$ . Note also that phase is not pictured nor mentioned explicitly. What is needed is stability and well-behaved crossover. For SISO systems, phase provides a very convenient way to evaluate stability and crossover properties but is not important in and of itself. Furthermore the phase of a rational function is completely determined by its gain, and location of rhp poles and zeros, as discussed in the companion paper. [19] Thus, for evaluating the feedback properties of a SISO control design, phase is not needed explicitly, given that stability is checked and plots of  $|1 + gk|$  and  $|1 + 1/gk|$  or  $|gk(1 + gk)^{-1}|$  are provided.

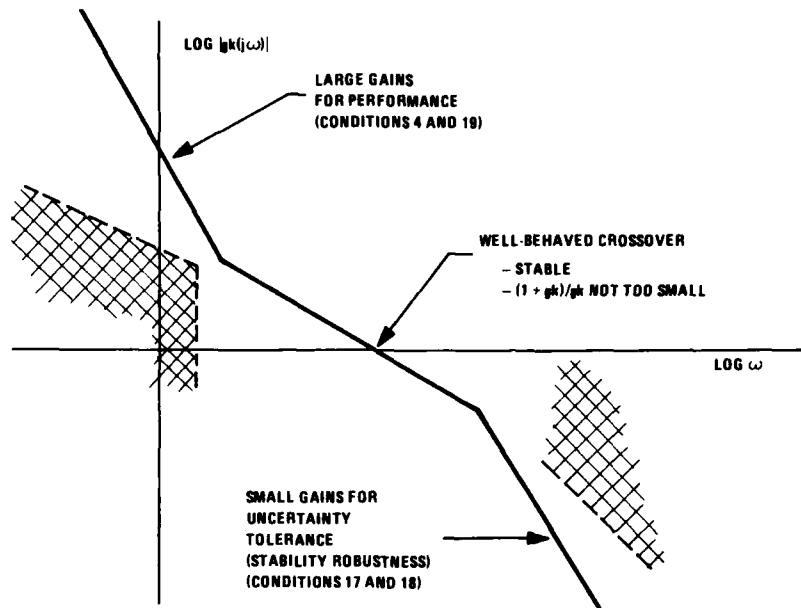


Figure 3. The Feedback Design Problem According to Bode

The  $\sigma$ -plot generalization of the Bode gain plot and its interpretation are displayed graphically in Figure 4. The performance conditions are essentially the same as for the Bode feedback design problem with singular values replacing complex magnitude. As in the SISO case phase is not needed. Thus the fact that there is no natural multivariable generalization of phase should not be disturbing, because the important feedback properties of a system do not depend explicitly on phase.

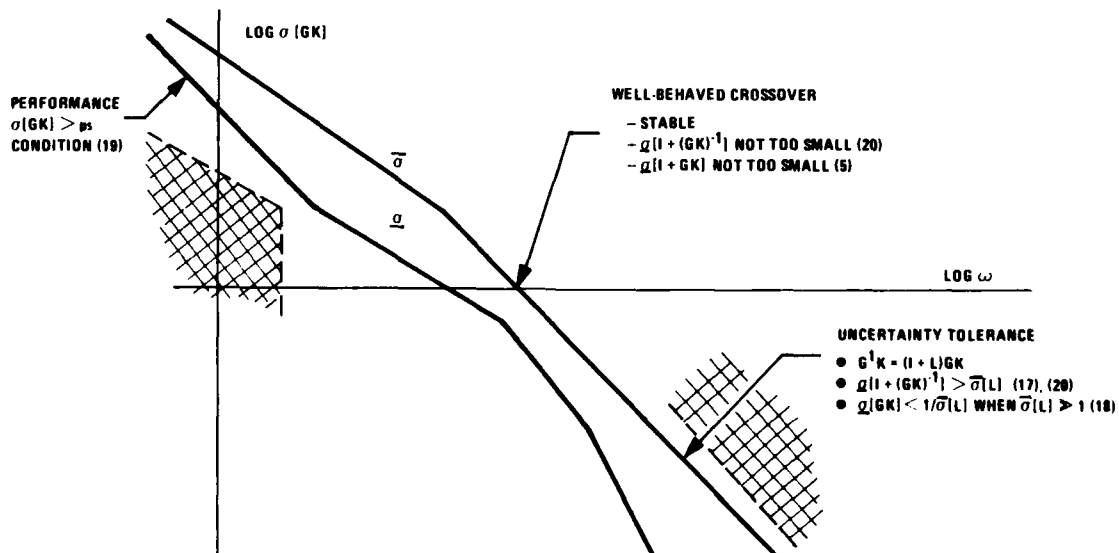


Figure 4. The Multivariable Feedback Design Problem

The  $\sigma$ -plots of a representative loop transfer matrix are also sketched in the figure. As in the SISO case the effective bandwidth of the loop cannot fall much beyond the frequency  $\omega_0$  for which  $I_m(\omega_0) = 1$ . As a result, the frequency range over which performance objectives can be met is explicitly constrained by the uncertainties. It is also evident from the sketch that the severity of this constraint depends on the rate at which  $g[GK]$  and  $\sigma[GK]$  are attenuated. The steeper these functions drop off, the wider the frequency range over which Condition (19) can be satisfied. Unfortunately, FDLTI transfer functions behave in such a way that steep attenuation comes only at the expense of small  $g[1 + GK]$  values and small  $g[1 + (GK)^{-1}]$  values when  $g[GK]$  and  $\sigma[GK] \approx 1$ . This means that while performance is good at lower frequencies and stability robustness is good at higher frequencies, both are poor near crossover. The behavior of FDLTI transfer functions, therefore, imposes a second major limitation on the achievable performance of feedback systems. These and other limitations on achievable performance will be considered in the companion paper [19].

## VI. Examples

Two examples will now be discussed to illustrate the ideas presented in the previous sections. Both examples are simple and of low order. Hopefully, this will encourage the reader to experiment with the examples to get a better understanding of the plots involved.

The first example is a two input oscillator with open loop poles at  $\pm 10j$  and both closed loop poles under unity feedback at  $-1$ . The loop transfer function is

$$G(s) = \frac{1}{s^2 + 100} \begin{bmatrix} s - 100 & 10(s + 1) \\ -10(s + 1) & s - 100 \end{bmatrix} \quad (21)$$

and with unity feedback,  $K = I$ . There are no transmission zeros.

Suppose we begin the analysis by breaking individual loops at the inputs and viewing the multiloop system as two single loop systems. With either loop closed (the system is symmetric) the transfer function for the other loop is

$$g(s) = \frac{1}{s}$$

which indicates that individually the loops have large classical SISO margins at crossover ( $\pm \infty$  db gain and  $90^\circ$  phase at  $\omega = 1$ ).

The singular values of  $GK(I + GK)^{-1}$  are plotted in Figure 5. From equation (17), we see that the peak in  $\sigma$  near 20 radians, indicates that there exists a small perturbation ( $\Delta_m = .1$ ) that would create an instability. Thus, while each loop individually may have large stability margins, the feedback system is extremely sensitive to perturbations which effect both loops.

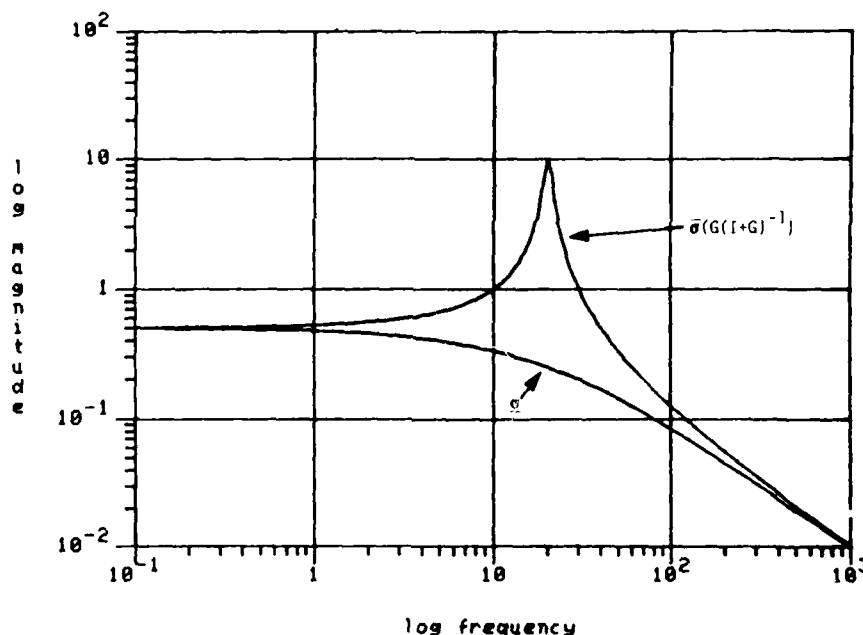


Figure 5.  $\sigma$ -Plot for example 1

To get further insight into this example, consider a diagonal perturbation (as in (13))

$$L = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad (22)$$

where  $k_1$  and  $k_2$  are real constants. The regions of stability and instability may be plotted in the  $(k_1, k_2)$  plane as shown in Figure 6. Gain-plane plots such as these have been used by engineers for decades, but they unfortunately do not easily generalize to higher dimensions.

In Figure 6, the open loop point is  $k_1 = k_2 = -1$  and the nominal closed loop point is  $k_1 = k_2 = 0$ . If each loop is broken individually, the  $k_1, k_2$  axes are checked for stability. As can be seen from Figure 6 this would miss the unstable region near the origin caused by simultaneous changes in  $k_1$  and  $k_2$ . Singular value analysis checks in all directions and gives a more reliable characterization of the robustness of a feedback system.





with  $\omega_R = 25$  rad/sec and conservatively confined to the range  $0.1 \leq \zeta \leq 1.0$ . Because collective and differential-collective inputs both involve coning motions of the rotors, one such transfer function will appear in each control channel.

The rotor dynamics can be represented as a multiplicative perturbation ( $L$  in Eq. (13)) where

$$L = (g_R - 1) I$$

and

$$l_m(\omega) = \max_{\zeta} \frac{s^2 + 2\zeta\omega_R}{s^2 + 2\zeta\omega_R s + \omega_R^2} \quad s = j\omega \quad (27)$$

This function was evaluated for a range of frequencies and is shown in Figure 7. From equations (17) and (18) we know that this uncertainty level will limit bandwidth to less than 10 rad/sec.

An alternative approach would be to explicitly add the rotor dynamics to the nominal model and represent the uncertain damping by a variable parameter, as in eq. (26). This would eliminate the perturbation due to the first mode of the rotor and allow for a higher bandwidth controller. Unfortunately, such a high-bandwidth controller would require modification and more accurate modeling of the blade angle actuation hardware, as well as additional modeling of higher-frequency rotor dynamics (second and higher harmonics). An engineer considering this option would have to weigh its cost against the benefits of a high bandwidth controller.

In any case, even with more accurate modeling and hardware improvements, the feedback designer would still be faced with large unstructured uncertainty, (possibly at higher frequencies,) due to neglected dynamics. This tradeoff, where more costly and accurate modeling allows for more accurate control, is a central issue in engineering modeling problems. By using the model of eq. (23) - (25), we have opted for modeling only the basic rigid body dynamics explicitly and including other effects in the unstructured uncertainty.

By combining eq. (27) with eqs. (17) - (19) we obtain the bounds on the singular values of the loop transfer function shown in Figure 8. The high frequency constraint is obtained from the neglected rotor dynamics in Figure 7. The constraint has been made linear for convenience, and continues downward beyond the 25 rad/sec peak because higher order harmonics and other uncertainties will dominate at these higher frequencies. We expect that second order attenuation characteristics beyond 25 rad/sec will be adequate to provide stability in the face of these additional uncertainties. The low frequency constraints should provide good command response and disturbance rejection as well as desensitizing the controller to changes in flight conditions.

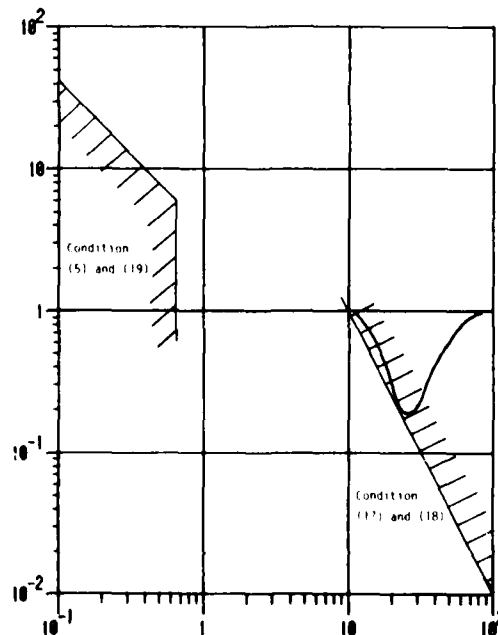
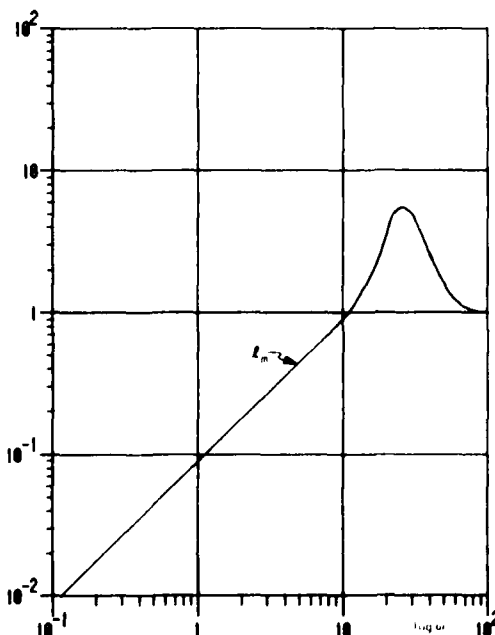


Figure 7. Uncertainty Bound for Rotor      Figure 8.  $\sigma$ -Plot Constraints for CH-47 Design

To strengthen the connection of the  $\sigma$ -plots with classical SISO techniques and Bode plot methods, we will first treat a single-loop problem of controlling pitch attitude ( $\theta$ ) with differential collective (dc). While this loop decouples from the vertical

velocity/collective loop at hover, they are highly coupled in the 40 knots forward flight condition so a single loop design would not be recommended here except for illustration purposes. The Bode gain plots of three alternative designs with loops broken at the input are shown in Figure 9. These controllers are all state feedback (plus dynamic compensation in case lc) since for the CH-47 all the states in the nominal model are measurable. The design technique used is described briefly in the Appendix.

Trial 1a controller violates the low frequency condition while Trial 1b violates the high frequency condition. By adding the second order rotor dynamics from (26) we can easily verify that Trial 1b is unstable for  $\omega = .1$ . Trial 1c is an intermediate design which includes a low-pass filter at  $\omega = 12$  rad/sec to help avoid the rotor uncertainty. Plots of  $|1 + g|$  and  $|g(1 + g)^{-1}|$  for Trial 1c are shown in Figure 10. Crossover is well-behaved since  $|1 + g|$  does not get too small and  $|g(1 + g)^{-1}|$  does not get too large. Recall that  $|g(1 + g)^{-1}|$  is the magnitude of the closed loop response or the feedback signal to the input. Thus there would be little peaking of the response. With the addition of Figure 10, Figure 9 is unnecessary, since the low and high frequency as well as crossover characteristics can be obtained from Figure 10 and equations (5) and (17).

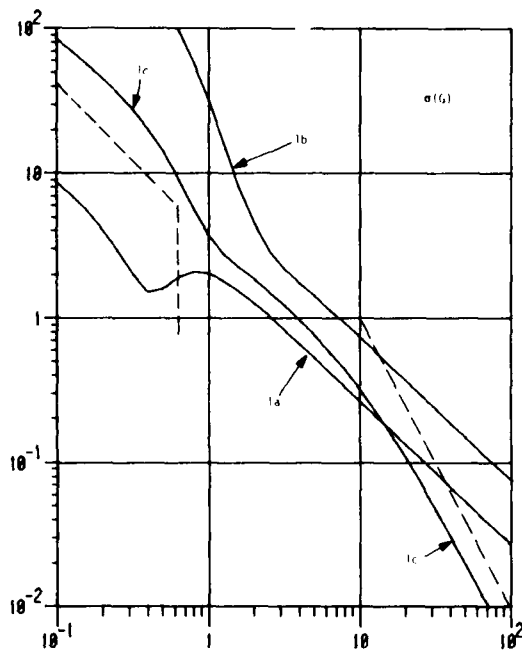


Figure 9. Bode Plots for SISO Design

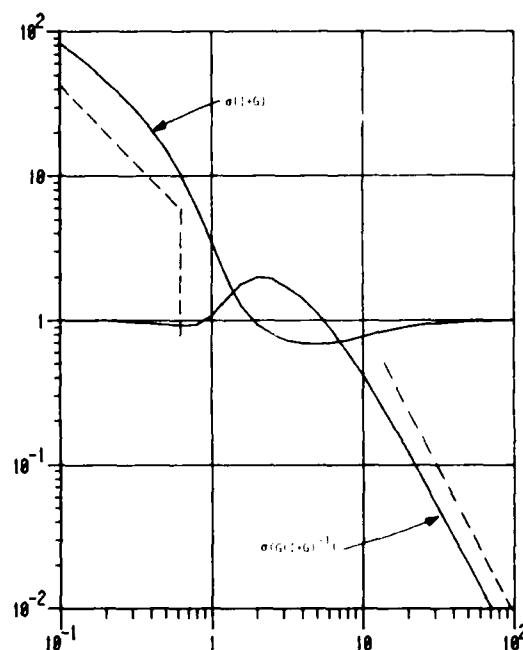


Figure 10. Bode Plots for SISO Trial 1c

The beauty of singular value analysis is that the above analysis for SISO systems carries over without change to MIMO systems. This is illustrated in Figures 11 and 12 with two trial two-input designs. The design technique used is described briefly in the Appendix. This distinction between Figures 11 and 12 and Figure 9 is that for the two-input designs, two plots are needed for each trial. Condition (18) implies that  $\sigma$  (and thus all  $\sigma$ ) must lie below the high frequency constraint and condition (19) implies  $\sigma$  (and thus all  $\sigma$ ) must lie above the low frequency constraint. Trial 2a (Figure 11) violates both conditions while Trial 2b (Figure 12) satisfies both. Both trials use state feedback but Trial 2b includes a first order lag at  $\omega = 12$  rad/sec to provide sufficient attenuation to avoid the uncertainty bounds. The  $\sigma$ -plots of  $(I + G)$  and  $G(I + G)^{-1}$  for Trial 2b are shown in Figure 13. These can be interpreted in terms of conditions (5) and (17) in the same way as for Trial 1c in Figure 10.

Which  $\sigma$ -plots an engineer chooses to use is somewhat a matter of taste and would depend in part on the design technique being used and on the stage of the design. Although we did not explicitly design for other flight conditions, the low frequency gain conditions provide considerable robustness with respect to flight condition variations. Trial 2b remains stable at eight representative flight conditions from hover to 160-knots forward speed and from -2000 ft/min to +2000 ft/min ascent rates. It must be emphasized, however, that these designs were for illustration purposes only and are not intended to be flight quality designs. Trial 2b might be considered as a candidate control law but would require implementation and verification through high-fidelity simulation and flight test before qualifying as a flight quality design.

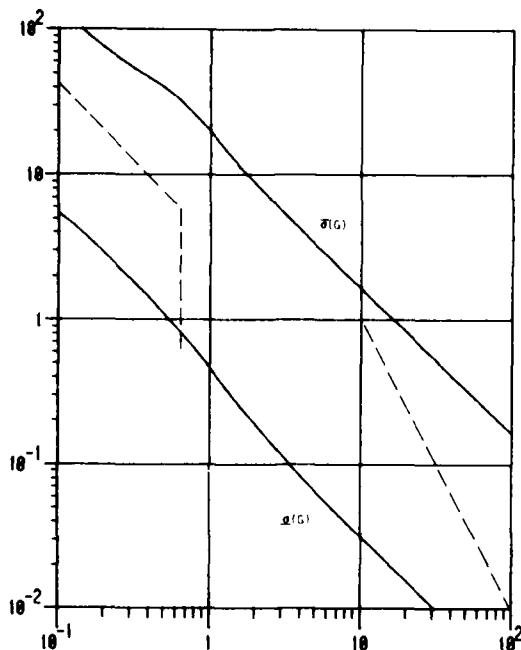
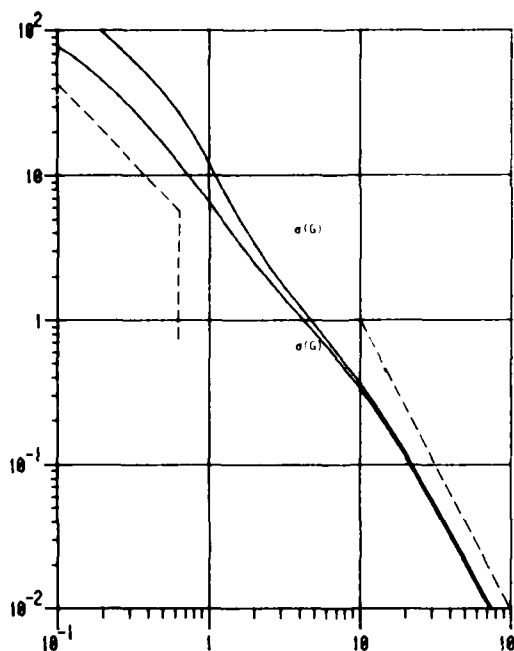
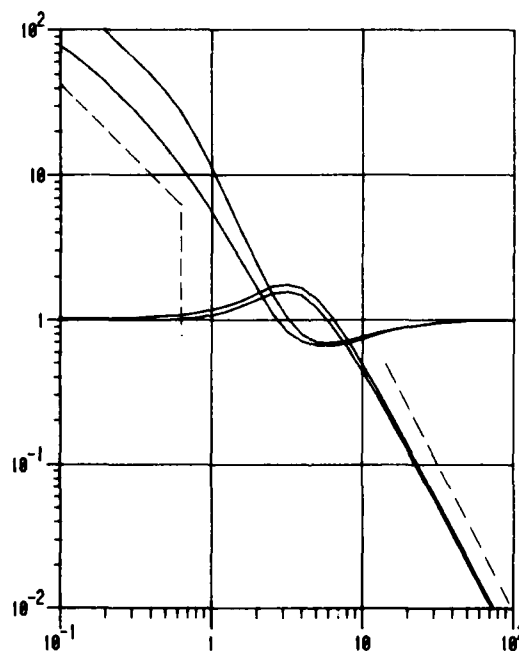
Figure 11.  $\sigma$ -Plots of  $G$  for Trial 2aFigure 12.  $\sigma$ -Plots of  $G$  for Trial 2b

Figure 13.

## VII. Conclusions

This paper has attempted to present a practical design perspective on multiloop linear time invariant feedback control problems. It has focused on the fundamental issue -- feedback in the face of uncertainties -- and has shown how classical SISO approaches to this issue can be reliably and meaningfully generalized to MIMO systems.

There were many other topics not considered in this paper which are less fundamental to the feedback problem but still very important. For example, we did not treat certain performance objectives in MIMO systems which are distinct from SISO systems. These include perfect non-interaction and integrity. Non-interaction is again a structural property which loses most of its meaning in the face of unstructured uncertainties. (It is achieved as well as possible by Condition 19). Integrity, on the other hand, cannot be dismissed as lightly. It concerns the ability of MIMO systems to maintain

stability in the face of actuator and/or sensor failures. The singular value concepts described here are indeed useful for integrity analysis. For example, a design has integrity with respect to actuator failures whenever

$$\sigma[I + (KG)^{-1}] > 1 \quad \forall \omega$$

This follows because failures satisfy  $\|L_m\| \leq 1$ .

The major limitations on what has been said in the paper are associated with the representation chosen in Section 4 for unstructured uncertainty. A single magnitude bound on matrix perturbations is a worst-case representation which is often much too conservative (i.e., it may admit perturbations which are structurally known not to occur). The use of weighted norms in (8) - (9) or selective transformations applied to  $G$  (as in [23]) can alleviate this conservatism somewhat, but seldom completely.

A related drawback is the implicit assumption that all loops (all directions) of the MIMO system should have equal bandwidth ( $\sigma$  close to  $\sigma$  in Figure 4). This assumption is consistent with a uniform uncertainty bound but is not appropriate for more complex uncertainty structures. These issues are important and complex enough to deserve more attention in their own right than can be given here. Research along these lines is proceeding.

This paper has not addressed the problem of actually synthesizing controllers to satisfy the design conditions developed. The approach taken here has been to examine fundamental feedback issues and provide reliable and effective tools for analyzing the feedback properties of multiloop systems. These issues are independent of the specific synthesis tools used and any technique which can provide a controller satisfying the constraints for a particular problem will be adequate. Unfortunately, most existing synthesis techniques are oriented at manipulating quantities which are at best only indirectly related to feedback properties. An exception to this is the technique described in [1] and in more detail in two papers by Stein ([24], [25]) in this lecture series. The reader is encouraged to consider the techniques described in these papers by Stein in the context of the feedback design problem as outlined in the preceding sections of this paper.

The companion paper [19] addresses the achievable performance of a feedback system in the face of uncertainty. The sources of limitations on achievable performance considered are:

- 1) The algebraic conflict between providing desensitization and insuring stability in the presence of large uncertainty.
- 2) The functional relationship between gain and phase for causal, rational transfer functions.
- 3) Right hand plane transmission zeros
- 4) Directionality conflicts

The last issue is a MIMO problem that has no SISO analogue while the first three are MIMO generalization of classical SISO properties.

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and

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#### APPENDIX

The nominal model for the dynamics of the CH-47 at 40 knots forward airspeed has [21]

$$A = \begin{bmatrix} -.02 & .005 & 2.4 & -32 \\ -.14 & .44 & -1.3 & -30 \\ 0 & .018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} .14 & -.12 \\ .36 & -8.6 \\ .35 & .009 \\ 0 & 0 \end{bmatrix}$$

for A and B in equation (23).

The example designs were all state feedback including integrators (and lag filters in Trials 1c and 2b). The state feedback gains were obtained using linear-quadratic methodology, selected primarily for its convenience and good frequency-domain properties (see [1], [24], [25]). In the context of feedback design, time-domain optimality, per se, is of no interest. The state feedback gains for Trail 2b (states 5 and 6 are integrators) are

$$K = \begin{bmatrix} -4.72 \times 10^{-4} & .0259 & 12.6 & 47.9 & .0417 & -57.2 \\ .0163 & -.535 & .231 & 2.16 & .999 & 2.39 \end{bmatrix}$$

Trial 2b also has first order lags

$$\frac{1}{1 + s/12}$$

added to each channel to provide additional high-frequency attenuation.

#### Acknowledgements

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# Limitations on Achievable Performance of Multivariable Feedback Systems

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## Summary

This paper examines the fundamental limitations on the achievable performance of multivariable feedback systems. Using the singular value analysis techniques discussed in the companion paper [1], the design tradeoffs that are the focus of classical single-loop theory are generalized to multiloop systems. The first tradeoff is algebraic between large loop gain for good sensitivity properties and small loop gain for stability margins. Another is the functional tradeoff imposed by the Bode gain/phase relations. These tradeoffs impose limitations on the achievable performance of any feedback design. The limitations caused by nonminimum phase zeros is also discussed. Finally, the problem of directionality in multiloop systems is presented and analyzed. This is a uniquely multiloop problem and has no analog in single loop feedback systems.

## 1. Introduction

Over the last fifteen years, a misconception has arisen which implies that the application of advanced control concepts can control any system to any level of desired performance. This misconception can be traced in part to the aura that originally surrounded optimal control and estimation theory. The notions that a design could be optimal, poles could be placed arbitrarily, and that variables that could not be measured could be estimated contributed heavily to the misconception. In reacting to the promises of optimal control and estimation, many designers lost sight of the true nature of the control problem. Control systems are designed to satisfy performance specifications. Performance is limited by the nature of the control application and the actuation and measurement devices available. Feedback control systems possess severe performance limitations because of the potential for destabilization and its consequences.

This paper examines limitations on the achievable performance of multivariable feedback systems. The notion of performance that is used in this paper is the desensitizing effect that feedback can have on uncertainty in the plant.

Section 2 reviews the fundamental properties of feedback systems involving uncertainty using the techniques presented in [1]. From this basis is developed the algebraic conflict between high desensitization and large stability margins. In Section 3, the Bode gain/phase relations are generalized to multivariable systems. These relations impose a functional tradeoff between the rate of attenuation of the loop gain and the stability margins at crossover. This result is used in Section 4 to study the impact on performance of transmission zeros in the right half plane.

In Section 5, the issue of directionality in multivariable feedback systems is explored. It is shown that under some circumstances there may be a conflict between margins at the input and at the output of the plant. Section 6 has the conclusions.

## 2. Algebraic Constraints on Feedback Systems

Consider the feedback configuration shown in Figure 1. This represents a (nominally) stable closed-loop system consisting of a linear time-invariant plant  $G(s)$  and controller  $K(s)$ . In this section, we will discuss multivariable generalizations of some classical properties of single-loop feedback systems.

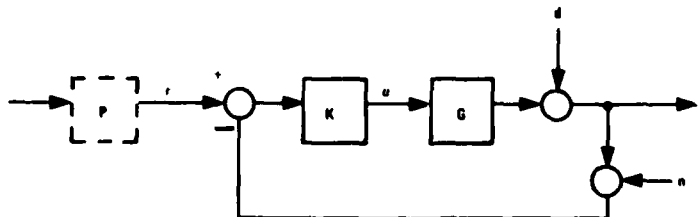


Figure 1.

One use of feedback is to reduce sensitivity to changes in the plant. It is well-known that a useful sensitivity function for a single-loop feedback system is the inverse of the return difference. To quantify the desensitizing effect of feedback in the multivariable case, we use the comparison sensitivity approach [4]. For the comparison, let  $G$  represent the nominal plant and  $G(I+\Delta)^{-1}$  represent the perturbed plant. To obtain a direct comparison of sensitivity with and without feedback, we consider the feedback system with precompensation  $P = (I+KG)$ . This precompensation is added so that the closed-loop transfer function for the nominal plant is identical with the nominal open-loop transfer function,  $G$ . The perturbed closed-loop transfer function is

$$G(I+\Delta)^{-1}(I+KG(I+\Delta))^{-1}(I+KG) = G(I+(I+KG)^{-1}\Delta)^{-1} \quad 2.1$$

$$G(I+\Delta_{CL})^{-1}$$

where the effective closed-loop perturbations  $\Delta_{CL}$  is

$$\Delta_{CL} = (I+KG)^{-1}\Delta \quad 2.2$$

Thus for multivariable systems, the inverse return difference matrix,  $(I+KG)^{-1}$ , relates the sensitivity of the closed-loop system to that of the open-loop system. In terms of the spectral norm, if the condition

$$1 > \bar{\sigma}[(I+KG)^{-1}] \quad 2.3$$

or, equivalently

$$\underline{\sigma}(I+KG) > 1 \quad 2.4$$

holds at some frequency, then the use of feedback accomplishes desensitization at that frequency.

Feedback can also reduce the effect of a disturbance,  $d$ , on the output,  $y$  (Figure 1). The response at the output to this disturbance is

$$y = (I+KG)^{-1}d \quad 2.5$$

Since the open-loop response to such a disturbance is simply  $d$ , satisfaction of condition (2.4) at some frequencies means that feedback provides some disturbance rejection at those frequencies.

In terms of sensitivity and disturbance rejection, good performance of a multivariable (as well as single-loop) feedback system translates into the requirement of a "large" return difference matrix. Since

$$|\underline{\sigma}(I+KG) - \underline{\sigma}(KG)| < 1 \quad 2.6$$

this is equivalent to requiring "large" loop gain. A more detailed discussion of the relationship between performance and loop gain may be found in the companion paper [1]. We will return to the desideratum of large loop gain after considering stability margins for multivariable feedback systems.

The representation of uncertainty as  $(I+\Delta)^{-1}$  is convenient for studying sensitivity. Some uncertainties in physical systems, however, cannot be represented in this way. In particular, both gain and phase become highly uncertain at high frequencies. Modeling the plant with uncertainty as  $G(I+\Delta)$  is quite useful in these situations [1].

Using this representation for the plant with uncertainty in the multivariable feedback system of Figure 1, the closed-loop system remains stable for all (stable) perturbations  $\Delta(s)$  satisfying

$$\bar{\sigma}(\Delta(j\omega)) < \underline{\sigma}[I+(K(j\omega)G(j\omega))^{-1}] \quad \forall \omega \geq 0 \quad 2.7$$

The size of  $I+(KG)^{-1}$  measures the robustness of closed-loop stability with respect to multiplicative perturbations  $(I+\Delta)$ . This is directly analogous to the distance to the critical point on an inverse Nyquist diagram,  $|1+1/KG|$ .

Adequate representations of physical systems inevitably require that the perturbation become large at high frequency, and so large  $(KG)^{-1}$  (or, equivalently, small loop gain  $KG$ ) is required at these frequencies to ensure stability.

This last statement leads to a trade-off in the design of feedback systems. At a particular frequency, it is possible to have large loop gain for desensitization and disturbance rejection or small loop gain for stability margins -- but not both. More precisely, consider the identity

$$(I+KG)^{-1} + ((I+(KG)^{-1})^{-1}) = I \quad 2.8$$

A direct consequence of this identity is the string of inequalities

$$\frac{\underline{\sigma}(I+L)}{\underline{\sigma}(I+L)+1} \leq \underline{\sigma}(I+L)^{-1} \leq \frac{\underline{\sigma}(I+L)}{\underline{\sigma}(I+L)-1} \quad 2.9$$



where either  $KG$  or  $(KG)^{-1}$  may be substituted for  $L$ , and the right-hand inequality requires  $\sigma(I+L) > 1$ . Clearly then,

$$\sigma(I+L) \gg 1 \rightarrow \sigma(I+L^{-1}) \approx 1$$

Thus, desensitization can be obtained only at frequencies where stability margins need not be large. Since large stability margins are always required at sufficiently high frequencies, no desensitization can be achieved there.

As an aside, note that the left-hand inequality in 2.9 may be rewritten as

$$\sigma(I+L) \leq \frac{\sigma(I+L^{-1})}{1-\sigma(I+L^{-1})} \quad 2.10$$

for  $\sigma(I+L^{-1}) < 1$ . Hence

$$\sigma(I+L^{-1}) \ll 1 \Rightarrow \sigma(I+L) \ll 1$$

Recall that  $L$  may be replaced by either  $KG$  or  $(KG)^{-1}$ . So roughly speaking, "poor" sensitivity properties at any frequency is equivalent to "poor" margins there.

### 3. Functional Limitations on Transfer Functions

We have seen that the trade-off between desensitization and stability margins applies to multivariable feedback systems as well as to single-loop systems. Typically, desensitization and disturbance rejection are desired at low frequencies, and large stability margins are required at high frequencies. A critical factor in the design process is the necessity to make the transition from large loop gain to small loop gain. Throughout this transition region, poor sensitivity and poor stability margins must be avoided.

The difficulty in this transition depends on the required attenuation rate of the loop gain. The steeper the loop gain drops off, the wider the frequency range over which it may be large. Unfortunately, however, FDLTI transfer functions behave in such a way that steep attenuation comes only at the expense of small  $\sigma[I+GK]$  values and small  $\sigma[I + (GK)^{-1}]$  values when  $\sigma[GK]$  and  $\sigma[GK] \approx 1$ . This means that while performance is good at lower frequencies and stability robustness is good at higher frequencies both are poor near crossover. The behavior of FDLTI transfer functions, therefore, imposes a second major limitation on the achievable performance of feedback systems.

#### SISO Transfer Function Limitation

For SISO cases, the conflict between attenuation rates and loop quality at crossover is again well understood. We know that any rational, stable, proper, minimum phase loop transfer function satisfies fixed integral relations between its gain and phase components. Hence, its phase angle near crossover (i.e., at values of  $\omega$  such that  $|gk(j\omega)| \approx 1$ ) is determined uniquely by the gain. Various expressions for this angle were derived by Bode using contour integration around closed contours encompassing the right half plane. [6, Chapters 13, 14] One expression is

$$\begin{aligned} \phi_{gkc} &= \Delta \arg[gk(j\omega_c)] \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln|gk(j\omega(v))| - \ln|gk(j\omega_c)|}{\sinh v} dv \end{aligned} \quad 3.1$$

where  $v = \ln(\omega/\omega_c)$ ,  $\omega(v) = \omega_c \exp v$ . Since the sign of  $\sinh(v)$  is the same as the sign of  $v$ , it follows that  $\phi_{gkc}$  will be large if the gain  $|gk|$  attenuates slowly and small if it attenuates rapidly. In fact,  $\phi_{gkc}$  is given explicitly in terms of weighted average attenuation rate by the following alternate form of (3.1) (also from [6]):

$$\phi_{gkc} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \ln|gk|}{dv} \left( \ln \coth \frac{|v|}{2} \right) dv \quad 3.2$$

The behavior of  $\phi_{gkc}$  is significant because it defines the magnitudes of our two SISO design conditions 2.2 and 2.7 at crossover. (Equations 17 and 19 in [1]). Specifically when  $|gk| = 1$ , we have

$$|1+gk| \approx |1+(gk)^{-1}| = 2 \left| \sin \left( \frac{\pi + \phi_{gkc}}{2} \right) \right| \quad 3.3$$

The quantity of  $\pi + \phi_{gkc}$  is the phase margin of the feedback system. Assuming  $gk$  stable, this margin must be positive for nominal stability and, according to 3.3 it must be reasonably large ( $\approx 1$  rad) for good return difference and stability robustness properties. If  $\pi + \phi_{gkc}$  is forced to be very small by rapid gain attenuation, the feedback system will amplify disturbances ( $|1+gk| \ll 1$ ) and exhibit little uncertainty tolerance at and near  $\omega_c$ . The conflict between attenuation rate and loop quality near cross-over is thus clearly evident.

It is also known that more general nonminimum phase and/or unstable loop transfer functions do not alleviate this conflict. If the plant has right half plane zeros, for example, it may be factored as

$$g(s) = m(s) p(s) \quad 3.4$$

where  $m(s)$  is minimum phase and  $p(s)$  is an all-pass (i.e.,  $|p(j\omega)| = 1 \forall \omega$ ). The (negative) phase angle of  $p(s)$  reduces total phase at crossover, i.e.,

$$\phi_{gkc} = \phi_{mkc} + \phi_{pc} \leq \phi_{mkc} \quad 3.5$$

and therefore aggravates the tradeoff problem. In fact, if  $|\phi_{pc}|$  is too large, we will be forced to reduce the crossover frequency. Thus rhp zeros limit loop gain (and thus performance) in a way similar to the unstructured uncertainty. A measure of severity of this added limitation is  $|1 - p(j\omega)|$ , which can be used just like  $\phi_m(\omega)$  to constrain a nominal minimum phase design..

If  $g(s)$  has rhp poles, the extra phase lead contributed by these poles compared with their mirror images in the left half-plane is needed to provide encirclements for stability. Unstable plants thus also do not offer any inherent advantage over stable plants in alleviating the cross-over conflict.

#### Multivariable Generalization

The above transfer function limitations for SISO systems have multivariable generalizations, with some complications as would be expected.

The major complication is that singular values of rational transfer matrices, viewed as functions of the complex variable  $s$ , are not analytic and therefore cannot be used for contour integration to derive relations such as 3.1. Eigenvalues of rational matrices, on the other hand, have the necessary mathematical properties. Unfortunately, they do not in general relate directly to the quality of the feedback design. Thus, we must combine the properties of eigenvalues and singular values through the bounding relations

$$\sigma[A] \leq |\lambda[A]| \leq \sigma[A] \quad 3.6$$

which holds for any eigenvalue,  $\lambda_i$ , of the (square) matrix  $A$ . The approach will be to derive gain/phase relations as in equation 3.1 for the eigenvalues of  $I + GK$  and  $I + (GK)^{-1}$  and to use these to bound their minimum singular values. Since good performance and stability robustness requires singular values of both of these matrices to be sufficiently large near crossover, the multivariable system's properties can then be no better than the properties of their eigenvalue bounds.

Equations for the eigenvalues themselves are straightforward. There is a one-to-one correspondence between eigenvalues of  $GK$  and eigenvalues of  $I + GK$  such that

$$\lambda_i[I+GK] = 1 + \lambda_i[GK] \quad 3.7$$

Likewise for  $I + (GK)^{-1}$ ,

$$\lambda_i[I+(GK)^{-1}] = 1 + \frac{1}{\lambda_i[GK]} \quad 3.8$$

Thus when  $|\lambda_i[GK]| = 1$  for some  $\lambda_i$  and  $\omega = \omega_c$ , we have

$$|\lambda_i[I+GK]| = |\lambda_i[I+(GK)^{-1}]| = 2 \left| \sin \frac{\pi + \phi_{\lambda_i c}}{2} \right| \quad 3.9$$

Since this equation is exactly analogous to equation 3.3 for the scalar case, and since  $|\lambda_i|$  bounds  $\sigma$ , it follows that the loop will exhibit poor properties whenever the phase angle  $(\pi + \phi_{\lambda_i c})$  is small.

In order to derive expressions for the angle  $\phi_{\lambda_i c}$  itself, we require certain results from the theory of algebraic functions. [9-15] The key concepts needed from these references are that the eigenvalues  $\lambda_i$  of a rational, proper transfer function matrix, viewed as a function of the complex variable  $s$ , constitute one mathematical entity,  $\lambda$ , called an algebraic function. Each eigenvalue,  $\lambda_i$ , is a branch of this function and is defined on one sheet of an extended Riemann surface domain. On its extended domain an algebraic function can be treated as an ordinary meromorphic function whose poles and zeros are the system poles and transmission zeros of the transfer function matrix. It also has additional critical points, called branch points, which correspond to multiple eigenvalues. Contour integration is valid on the Riemann surface domain provided that contours are properly closed.

In the contour integral leading to 3.1,  $g_k(s)$  may therefore be replaced by the algebraic function,  $\lambda(s)$ , with contour taken on its Riemann domain. Carrying out this integral yields several partial sums:

$$\sum_i \phi_{\lambda_i c} = \frac{1}{\pi} \int_{-\infty}^{\infty} \sum_i \frac{\ln |\lambda_i(j\omega(v))| - \ln |\lambda_i(j\omega_c)|}{\sinh v} dv \quad 3.10$$

where each sum is over all branches of  $\lambda(s)$  whose sheets are connected by right half-plane branch points. Thus the eigenvalues  $\{\lambda_i\}$  are restricted in a way similar to scalar transfer functions but in summation form. The summation, however, does not alter the fundamental tradeoff between attenuation rate and loop quality at crossover. In fact, if we deliberately choose to maximize the bound 3.9 by making  $\omega_c$  and  $\phi_{\lambda_i c}$  identical for all  $i$ , then 3.10 imposes the same restrictions on multivariable loops as 3.1 imposes on SISO loops. Hence, multivariable systems do not escape the fundamental transfer function limitations.

As in the scalar case, expression 3.10 is again valid for minimum phase systems only. That is GK can have no transmission zeros in the rhp. [For our purposes, transmission zeros [17] are values such that  $\det [G(s)K(s)] = 0$ . Degenerate systems with  $\det [GK] = 0$  for all  $s$  are not of interest because they cannot meet the condition in equation (2.4) at any frequency.] If this is not true, the tradeoffs governed by 3.9-3.10 are aggravated because every rhp transmission zero adds the same phase lag as in 3.5 to one of the partial sums in 3.10. The matrix GK may also be factored, as in 3.4 to get

$$GK(s) = M(s) P(s) \quad 3.11$$

where  $M(s)$  has no rhp zeros and  $P(s)$  is an all-pass matrix  $P^T(-s)P(s) = I$ . Analogous to the scalar case,  $\sigma(I-P(s))$  can be taken as a measure of the degree of multivariable nonminimum phaseness and used like  $\ell_m(\omega)$  to constrain a nominal minimum phase design. Nonminimum phase systems are considered in more detail in the next section.

#### 4. Non-Minimum Phase Behavior in Multivariable Systems

Given a desired rational, strictly proper loop transfer function  $H(s)$  that yields a stable closed-loop system, there exists rational, strictly proper compensation  $K(s)$  for the minimum phase plant  $G(s)$  such that the desired loop transfer function is fit arbitrarily closely and the resulting closed-loop system is stable. Indeed, it suffices to introduce the feedback compensation

$$K(s) = G^{-1}(s)H(s) \left( \frac{\alpha}{s+\alpha} \right)^m \quad 4.1$$

where  $m$  is chosen to make  $K(s)$  strictly proper and  $\alpha$  is chosen sufficiently large. This means that if  $H(s)$  is designed to achieve the sensitivity and stability margin tradeoff discussed in Section 2.0, then any minimum phase plant can be compensated to realize this loop transfer function as closely as desired. The same result applies to multivariable systems as well as single-loop systems by interpreting (4.1) in a matrix sense. In the multivariable case, however, such compensation yields the desired loop properties only at the plant inputs; the output loop properties can be quite different.

This is not true for non-minimum phase systems. A transfer function with right-half-plane zeros may be factored as

$$G(s) = M(s)P(s) \quad 4.2$$

where  $M(s)$  is minimum phase and  $P(s)$  is an all-pass ( $|P(j\omega)| = 1, \forall \omega$ ). The all-pass cannot be cancelled as in (4.1). It contributes a negative phase angle and thereby aggravates the tradeoff problem. Roughly speaking, a non-minimum phase zero  $\lambda$  requires small loop gain throughout some frequency interval "near"  $\lambda$  [7]. That is, the advantages of feedback are achievable at frequencies below and above the non-minimum phase zero's magnitude but not around it.

Multivariable non-minimum phase systems are subject to analogous restrictions. A plant transfer matrix with right-half-plane zeros may be factored in two ways as

$$G(s) = M_1(s)P_1(s) = P_2(s)M_2(s) \quad 4.3$$

where the  $P$ 's are all-pass ( $P^*(j\omega)P(j\omega) = I \forall \omega$ ) and the  $M$ 's are minimum phase. These factorizations can be expressed in state-space terms. Let  $G(s)$  be the transfer matrix of the quadruple  $(A, B, C, D)$ . Then for our purposes,  $G(s)$  has a zero at  $\lambda$  means

$$\begin{bmatrix} \lambda I - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \quad 4.4$$

has a solution with  $w^*w = 1$  [17] for a single real zero at  $\lambda$ . One factorization (unique up to a unitary transformation)  $M_1(s)P_1(s)$  is given by

$$P_1(s) = I - \left( \frac{2\lambda}{s+\lambda} \right) w w^T \quad 4.5$$

and  $M_1(s)$  is the transfer matrix of the quadruple  $(A, B, C, D)$  where

$$B = B - 2\lambda x w^T \quad 4.6$$

For the case of a single pair of complex zeros at  $\lambda$  and  $\bar{\lambda}$

$$P_1(s) = I - \left( \frac{4\alpha\gamma}{s^2 + 2\alpha s + \lambda\bar{\lambda}} \right) [s \operatorname{Re}(w w^* - \frac{\alpha\beta}{\lambda} \bar{w} w^*) + \operatorname{Re}(\lambda w w^* - \alpha\beta \bar{w} w^*)] \quad 4.7$$

$$\hat{B} = B - 2\alpha\gamma [x w^* - \frac{\alpha\beta}{\lambda} \bar{x} w^*] \quad 4.8$$

where  $\alpha = \operatorname{Re}(\lambda)$ ,  $B = w^T w$ , and  $\gamma = (1 - \frac{\alpha^2 \beta^2}{\lambda \bar{\lambda}})^{-1}$ . The proofs of (4.5) - (4.8) follow from the generalized eigenvalue problem (4.4) with algebraic manipulation. Multiple non-minimum phase zeros can be handled by repeated application of (4.5) - (4.8). We note that similar formulas may be obtained for the factorization  $P_2(s)M_2(s)$  by using the left generalized eigenvector.

$$\begin{bmatrix} \hat{x}^T & v^T \end{bmatrix} \begin{bmatrix} \lambda I - A & -B \\ -C & -D \end{bmatrix} = 0 \quad 4.9$$

The non-minimum phase zero  $\lambda$  adds extra phase lag in the input direction  $w$  and the output direction  $v$ . This is the same effect and has the same restrictions for feedback design as in the single-loop case. However, the transfer matrix from inputs orthogonal to  $w$  to outputs orthogonal to  $v$  does not have any additional phase lag caused by  $\lambda$ . Thus, the non-minimum zero  $\lambda$  does not impose any limitations on the loop gain in these orthogonal directions. [Other considerations may restrict this loop gain. For example, the transfer matrix associated with this subspace may itself be non-minimum phase, although not having a zero at  $\lambda$ .]

Because a non-minimum phase zero  $\lambda$  does not impose limitations in directions orthogonal to  $w$  and  $v$ , it could be the case that a feedback design has large loop gain in these directions. The next section discusses problems associated with a spread in the singular values of KG. Such a design would necessarily have a large spread in the singular values of KG at frequencies near  $\lambda$ .

## 5. Directionality in Multivariable Feedback Systems

The preceding sections discussed frequency-domain tradeoffs which occur in the design of both single-loop and multivariable feedback systems. In this section, we examine some tradeoffs that are unique to multivariable systems.

Multivariable system, unlike scalar system, signals have a "spatial" as well as frequency distribution. Some sensors are noisier than others. Certain actuators might saturate at lower signal levels than others. Disturbances enter particular loops, but not others, etc. One is lead to consider, for example, different "bandwidths" in various loops.

The stability condition (2.7) is tight if all that is known about the perturbation  $\Delta$  is its maximum singular value. However, if more is known about  $\Delta$  - large in one direction, small in another -- then condition (2.7) can be sharpened through the use of frequency-dependent weighting matrices [8]. That is, for a perturbed plant  $G(s)(I + \Delta(s))$ , the closed-loop system remains stable for all (stable) perturbations  $\Delta(s)$  satisfying

$$\sigma[Q_I(j\omega)\Delta(j\omega)R_I^{-1}(j\omega)] < \sigma[R_I(j\omega)(I + K(j\omega)G(j\omega))^{-1}Q_I^{-1}(j\omega)] \quad \forall \omega > 0 \quad 5.1$$

The known characteristics of the allowable perturbations are reflected in the (nonsingular) weighting matrices  $R_I$  and  $Q_I$ .

The stability margin condition (5.1) yields margins at a particular point in the feedback configuration of Figure 1. These margins are at the inputs to the plant. The weighting matrices in (5.1) are useful in handling the issue of directionality at this point. For single-loop systems, margins at the input are identically equivalent to margins at the output. Input and output margins are not equivalent, however, in the multivariable case. Output margins are obtained by considering a perturbed plant  $(I + \Delta_O(s))G(s)$ . The stability condition (using weighting matrices  $R_O$  and  $Q_O$ ) may be written as

$$\sigma[Q_O(j\omega)\Delta_O(j\omega)R_O^{-1}(j\omega)] < \sigma[R_O(j\omega)(I + (G(j\omega)K(j\omega))^{-1}(j\omega))] \quad \forall \omega > 0 \quad 5.2$$

This is the same condition as (5.1), with  $G$  and  $K$  interchanged.

From a design viewpoint, good margins may be desired at both the inputs and the outputs of the plant. One may express this objective slightly differently by first fixing the input margins and then examining the limits imposed on the output margins. More precisely, we pose the following formal optimization problem at a single frequency:

$$\begin{aligned} \max_K & \sigma[R_O(I + (GK)^{-1})Q_O^{-1}] \\ \text{s.t. } & \sigma_i[R_I(I + (KG)^{-1})Q_I^{-1}] = 1 \quad \forall i \end{aligned} \quad (P)$$

and now we require both  $G$  and  $K$  to be square.

This constraint of all singular values equal to 1 is equivalent to constraining the matrix to be unitary. Hence

$$I + (KG)^{-1} = R_I^{-1} U Q_I \quad 5.3$$

for some unitary  $U$ . The function to be maximized can then be written as

$$\begin{aligned} \sigma[R_O(I + (KG)^{-1})Q_O^{-1}] &= \sigma[R_O G(I + (KG)^{-1})G^{-1}Q_O^{-1}] \\ &= \sigma[R_O G R_I^{-1} U (Q_O G Q_I^{-1})^{-1}] \end{aligned} \quad 5.4$$

This means the original problem (P) is equivalent to

$$\begin{aligned} \max_U & \sigma[R_O G R_I^{-1} U (Q_O G Q_I^{-1})^{-1}] \\ \text{s.t. } & U U^* = I \end{aligned} \quad (P^1)$$

To solve  $(P^1)$ , let  $R_O G R_I^{-1} = U_1 \Sigma_1 V_1^*$  and  $Q_O G Q_I^{-1} = U_2 \Sigma_2 V_2^*$  be singular value decompositions where

$$\Sigma_1 = \text{diag}[\alpha_1, \alpha_2, \dots, \alpha_n] \quad , \quad \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$$

$$\Sigma_2 = \text{diag}[\beta_1, \beta_2, \dots, \beta_n] \quad , \quad \beta_1 \geq \beta_2 \geq \dots \geq \beta_n \geq 0$$

Making these substitutions into  $(P^1)$ ,

$$\sigma[R_O G R_I^{-1} U (Q_O G Q_I^{-1})^{-1}] = \sigma[\Sigma_1 \hat{U} \Sigma_2^{-1}] \quad \text{where } \hat{U} = V_1^* U V_2 \text{ is a unitary matrix.} \quad 5.5$$

#### Proposition

$$\begin{aligned} \max_U & \sigma[\Sigma_1 \hat{U} \Sigma_2^{-1}] = \min \left( \frac{\alpha_i}{\beta_i} \right) \\ \text{s.t. } & \hat{U} \hat{U}^* = I \end{aligned}$$

proof: Appendix

The formal problem (P) can be interpreted in terms of this proposition. The best possible output margins equal the minimum of the ordered ratio of the singular values of  $R_O G R_I^{-1}$  to the singular values of  $Q_O G Q_I^{-1}$ . When the uncertainty at the input and output is unstructured, i.e.,  $R_I, R_O, Q_I, Q_O$  are all scalar multiples of the identity, the maximum achievable margins at the outputs equal 1 -- the same as the margins at the inputs. This unstructured case causes the multivariable problem to be analogous to the single-loop case. When the uncertainties do have some directionality or structure (as reflected in non-identity  $R$ 's and  $Q$ 's), the feedback problem is inherently multivariable. This directionality may impose additional limitations on the achievable performance.

In the Appendix, it is shown that using  $U$  equal to the identity in the proposition achieves the maximum. With this choice of  $U$ , we can solve for  $K$  from (5.3) as

$$K = (R_I^{-1} V_1 V_2^* Q_I - I)^{-1} G^{-1} \quad 5.6$$

assuming that  $U = I$  yields  $R_I^{-1} V_1 V_2^* Q_I - I$  invertible. As far as actually achieving this limit at each frequency, note that  $V_1$  and  $V_2^*$  need not be rational functions of frequency. Hence, in general, we cannot achieve this limit with a causal compensator  $K(s)$ . The question of the limitations imposed on the output margins by both fixed input margins and a causal compensator is still open.

In concluding this analysis, we point out that the formal problem considered here is not especially important in and of itself. One could pose several other such problems, e.g., maximizing  $\sigma(I + KG)$  subject to constraints on  $I + GK$ , etc. The reason for posing this problem is that it illustrates how directionality can introduce uniquely multivariable phenomena.

Recall from the section on multivariable nonminimum phase systems that a rhp zero may cause a spread in singular values. This may cause a problem similar to that caused by nonidentity  $R$  and  $Q$  weightings. Thus, non-minimum phase zeros affect multivariable feedback design in two ways. First, they contribute additional phase lag in certain input and output directions. This limits the achievable performance in these directions and is analogous to non-minimum phase zeros in single-loop systems. The resulting unequal gains in different directions can lead to a conflict between input and output loop properties. This second effect has no single-loop analog.

A related problem arises when simultaneous input and output margins are considered. This involves letting the perturbed plant be  $(I + \Delta_1)G(I + \Delta_2)$  where  $\Delta_1$  and  $\Delta_2$  are independent perturbations. The companion paper [1] has an example indicating that evaluating margins one channel at a time does not give a reliable measure of robustness with respect to simultaneous variations in multiple channels. Thus it is not surprising that simple examples can be constructed where robustness with respect to simultaneous input and output perturbations is significantly worse than robustness for either point considered separately. If a realistic representation of the plant uncertainty involved perturbations at both inputs and outputs, then caution should be used in drawing conclusions based on analysis for either point individually.

To analyze the problem of simultaneous perturbations we may consider a unity feedback system with nominal transfer function

$$\begin{bmatrix} 0 & -G \\ K & 0 \end{bmatrix} \quad 5.7$$

and perturbed transfer function

$$\begin{bmatrix} 0 & -G \\ K & 0 \end{bmatrix} \begin{bmatrix} I+\Delta_1 & 0 \\ 0 & I+\Delta_2 \end{bmatrix} \quad 5.8$$

This system may then be analyzed for robustness with respect to the perturbation

$$\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \quad 5.9$$

using the techniques outlined in Section 2. Unfortunately, this would, in general, give a conservative answer since the standard singular value analysis tools do not exploit the structure of the perturbation in (5.9). Again, simple examples can easily be constructed to demonstrate this.

Recent research has led to an exact solution (i.e., not conservative) for perturbations with structure such as in (5.9). These results are as yet unpublished and are too lengthy for inclusion here. They provide reliable and exact techniques for analyzing simultaneous independent perturbations and generalized the ideas that are presented in this paper. The point that must be emphasized is that while the issues presented in this paper are central to the multivariable control problem, the results should not be interpreted too broadly.

## 6. Conclusions

This paper has used singular value analysis techniques to study some limitations on the achievable performance of multivariable feedback systems. The results give useful insight into the fundamental tradeoffs that must be faced in feedback design. It is important, though, that these results not be interpreted too broadly.

In particular, the development in this paper is often quite qualitative. While there is a central theme of feedback as a means of dealing with uncertainty, the specific results are fragmented. A unified framework which deals naturally with the algebraic and functional performance limitations would be highly desirable. In addition, the issue of robustness with respect to simultaneous input and output perturbations needs further study, as does the more general problem of analyzing perturbations with specific structure. Great progress is being made in these research areas.

The important message in this paper is that feedback design inevitably involves tradeoffs. Analysis and design techniques must make these tradeoffs clear to the control engineer so that there is a rational basis for making design decisions.

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## APPENDIX

Proposition: Let  $\varepsilon_1 = \text{diag}[a_1, a_2, \dots, a_n]$ ,  $a_1 \geq a_2 \geq \dots \geq a_n$  and

$\varepsilon_2 = \text{diag}[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n]$ ,  $\varepsilon_1 \geq \varepsilon_2 \geq \dots \geq \varepsilon_n$ . Then

$$\max_{\hat{U}} \sigma[\varepsilon_1 \hat{U} \varepsilon_2^{-1}] = \min_i \left( \frac{a_i}{\varepsilon_i} \right)$$

s.t.  $\hat{U} \hat{U}^* = I$

Proof: For any unitary  $\hat{U}$ ,

$$\begin{aligned} \sigma[\varepsilon_2 \hat{U}^* \varepsilon_1^{-1}] &= \sigma[\varepsilon_2 \hat{U}^* \varepsilon_1^{-1}] \\ &\geq \frac{\varepsilon_i}{a_i} \cdot \underbrace{\text{diag}[1, \dots, 1, 0, \dots, 0]}_{i \text{ times}} \hat{U}^* \underbrace{\text{diag}[0, \dots, 0, 1, \dots, 1]}_{n-i+1 \text{ times}} \end{aligned} \quad (\text{A.1})$$

where  $\frac{\varepsilon_i}{a_i} = \max_j (\varepsilon_j / a_j)$ . By compatibly partitioning  $\hat{U}$ , this may be written as

$$\begin{aligned} \sigma[\varepsilon_2 \hat{U}^* \varepsilon_1^{-1}] &\geq \frac{\varepsilon_i}{a_i} \cdot \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{U}_1^* & \hat{U}_4^* & \hat{U}_7^* \\ \hat{U}_2^* & \hat{U}_5^* & \hat{U}_8^* \\ \hat{U}_3^* & \hat{U}_6^* & \hat{U}_9^* \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| \\ &= \frac{\varepsilon_i}{a_i} \cdot \left| \begin{pmatrix} 0 & \hat{U}_4^* & \hat{U}_7^* \\ 0 & \hat{U}_5^* & \hat{U}_8^* \\ 0 & 0 & 0 \end{pmatrix} \right| \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \text{But } \begin{pmatrix} 0 & 0 & 0 \\ \hat{U}_4 & \hat{U}_5 & 0 \\ \hat{U}_7 & \hat{U}_8 & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{U}_4^* & \hat{U}_7^* \\ 0 & \hat{U}_5^* & \hat{U}_8^* \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - \hat{U}_6 \hat{U}_6^* & -\hat{U}_6 \hat{U}_9^* \\ 0 & -\hat{U}_9 \hat{U}_6^* & I - \hat{U}_9 \hat{U}_9^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \hat{U}_6 \\ \hat{U}_9 \end{pmatrix} \begin{pmatrix} 0 & \hat{U}_6^* & \hat{U}_9^* \end{pmatrix} \end{aligned} \quad (\text{A.3})$$

Hence the norm of the matrix on the rhs of (A.2) is one, and so

$$\sigma[\varepsilon_2 \hat{U}^* \varepsilon_1^{-1}] \geq \frac{\varepsilon_i}{a_i} \quad (\text{A.4})$$

Equation (A.4) is true for all unitary  $\hat{U}^*$ , with equality being obtained for  $\hat{U}^* = I$ . So

$$\max_{\hat{U}} \sigma[\varepsilon_1 \hat{U} \varepsilon_2^{-1}] = \left( \min_{\hat{U}} \sigma[\varepsilon_2 \hat{U}^* \varepsilon_1^{-1}] \right)^{-1} = \min_i \left( \frac{a_i}{\varepsilon_i} \right)$$

Q.E.D.

# LQG-BASED MULTIVARIABLE DESIGN: FREQUENCY DOMAIN INTERPRETATION

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## Summary

This paper demonstrates how the linear-quadratic-gaussian (LQG) methodology can be used to design feedback compensators which meet multivariable performance, stability, and stability robustness requirements expressed as singular value conditions in the frequency domain.

## 1.0 Introduction

For more than two decades now, feedback control design has been viewed from two distinct perspectives. One perspective is based on the frequency domain. Its roots lie in the classical single-input single-output (SISO) design methodologies of Bode and Nyquist [eg. 6,7], and its philosophy has been carried over to various attempted multivariable generalizations [eg. 9, 10, 11]. The second perspective is based on the time domain. Its roots lie in the state space concept for system descriptions and in linear-quadratic-gaussian (LQG) optimization for feedback synthesis [eg. 13, 14].

Despite recognized one-to-one mathematical relations between these two perspectives, feedback design has been taught and practiced in terms of one perspective or the other. Advocates developed for each side, and the pros and cons were debated enthusiastically. The frequency domain claimed to provide a natural framework, namely transfer function loop-shaping, in which to express and satisfy practical design requirements, particularly the ever-present requirement for uncertainty tolerance. This natural framework, however, worked best for SISO systems and did not appear to generalize easily to multi-input multi-output (MIMO) problems. The time domain claimed precisely the latter capability, i.e., its design techniques handled MIMO systems as well as time variations and stochastics with efficient computerized algorithms. Its weakness, however, was the lack of clear relationships between practical design requirements and the postulated linear-quadratic-gaussian notion of optimality.

Fortunately, the last few years have brought a synthesis of these perspectives which capitalizes on the strengths of both [1]. The first step of this synthesis was the reliable extension of classical frequency domain loop-shaping ideas to MIMO problems. This step is discussed in [2] earlier in these proceedings. The second step of the synthesis was the discovery that desirable multivariable loop-shapes can be produced with the linear-quadratic-gaussian methodology. This step is described in this paper. We begin with a brief review of the multivariable design problem in Section 2. We then show how this problem can be addressed with the LQG methodology in Section 3, and provide a simple illustrative example in Section 4. The bulk of this material is taken directly from Reference [1].

## 2.0 The Multivariable Feedback Design Problem

Our objective will be to design a dynamic compensator for the generic feedback loop shown in Figure 1. We have a plant ( $G$ ) with input vector ( $u$ ), output vector ( $y$ ), disturbance vector ( $d$ ) as seen at the output, and measurement noise vector ( $n$ ). Our compensator is designated as  $K$ . For convenience,  $K$  is shown in the generic forward path location from command vector ( $r$ ), i.e., unity feedback. More general situations with  $K$  in the return path from  $y$  or with  $K$  split between the forward and return paths are equivalent to the forward-path location if appropriate pre-filters ( $P$ ) are used to shape the commands. Since  $P$  does not affect feedback properties, however, we will not deal with its design issues here.

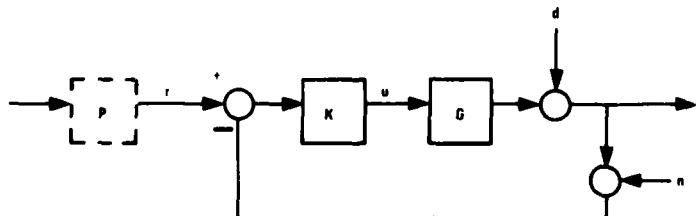


Figure 1.



The plant  $G$  is a real physical system. Hence, its detailed behavior must be expected to be quite complicated--nonlinear, infinite dimensional, time-varying, etc. Nevertheless, we will assume that  $G$  can be approximated by a finite dimensional, linear, time invariant (FDLTI) model with transfer function,  $G(s)$ . The inevitable errors which this model makes will be represented by unstructured multiplicative perturbations of the form [2]

$$G'(s) = [I + L(s)]G(s). \quad (1)$$

Here  $G'(s)$  is a possible "true" system, and  $L(s)$  is our FDLTI system with known bounded maximum gain,  $l_m(s)$ . Any  $L(s)$  which does not exceed this maximum gain generates another potential "true" system. Hence, equation (1) actually describes an entire family\* of plants for which our compensator must be designed.

The design problem will be to provide good command-following (i.e., small errors,  $e=y-r$ , in response to commands), good disturbance rejection (i.e., small errors in response to disturbance excitation), and small responses to sensor noise, all subject to the constraints imposed by modelling errors in (1). In an earlier paper in these proceedings [2], Doyle shows that these various design objectives can be interpreted as conditions on the loop transfer function matrices  $GK(s)$ , evaluated at real frequencies  $s=j\omega$ . We need large loop transfer matrices in the frequency range where commands and/or disturbances are large, small loop transfer matrices in the frequency range where modelling errors are large, and well-behaved crossover regions in between. "Well-behaved" crossovers are those which achieve nominal stability and maintain reasonably large return difference matrices,  $I+GK(j\omega)$ , and inverse return difference matrices,  $I+[GK(j\omega)]^{-1}$ . These frequency domain requirements of multivariable feedback design are summarized in Figure 2.

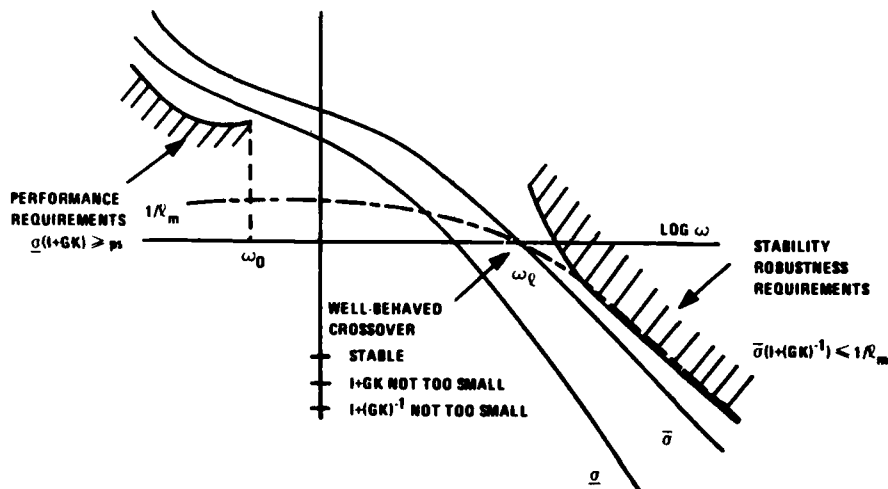


Figure 2.

Classical feedback designers will recognize that the frequency domain interpretations above are completely analogous to the classical single-input single-output design methodology [References 6, and 7 for example]. The only distinction, in fact, is that different measures are used to judge when transfer functions are "large" or "small." For scalar transfer functions, the absolute value serves as a single measure of size. The scalar  $gk$  is large if  $|gk| \gg 1$ , and it is small if  $|gk| \ll 1$ . For multivariable systems, on the other hand, two singular values (or principal gains) serve as measures of size. The matrix,  $GK$ , is large if its smallest singular value (smallest gain) is large, i.e., if  $\sigma[GK] \gg 1$ . It is small if its largest singular value (largest gain) is small, i.e.,  $\bar{\sigma}[GK] \ll 1$ . More discussion of singular values and their interpretations are given in [1], [2] and [8]. The key point to note here is that singular values reduce the multivariable feedback design problem, in concept, to the familiar frequency domain loop-shaping problem of classical control.

\*For deep technical reasons discussed in [3], the number of unstable poles of  $G'(s)$  must be known for all members of this family. We will assume that this number is constant and equal to the numbers of unstable poles of  $G(s)$ . Note that each  $G'(s)$  in the family is FDLTI. This restriction can be removed via "sector theory" [4, 5].

### 3.0 Singular-Value Loop-Shaping

The discussions in [2] concentrate on singular values as analysis tools, i.e., as instruments to specify, analyze and judge the quality of multivariable feedback loops. Little is said about their use as synthesis tools. It is clear, of course, that once we accept these instruments for analysis, then it also becomes desirable to have design techniques which directly manipulate them for synthesis. Ideally, we seek design techniques which take plant descriptions and performance specifications as inputs and automatically produce compensators as outputs which satisfy the various singular value requirements of Figure 2.

At today's state-of-the-art, there are no design techniques which fully satisfy these synthesis desires. There are a number of techniques, however, which offer multivariable design capability and which may be compared against the singular value loop-shaping interpretation. One major group of techniques, for example, consists of frequency domain methods which approximate the multivariable problem by a sequence of scalar problems. This is done by constructing a set of scalar design functions which may be manipulated more or less independently with classical techniques. The ad hoc "single-loop-at-a-time" technique in common engineering practice [9], for instance, uses a predefined sequence of inner, outer, and crossfeed loops to accomplish the design. The more systematic "Inverse Nyquist Array (INA)" procedure [10] uses diagonal elements of a pre- and post-compensated diagonally dominant transfer matrix as design functions, and the "Characteristic Loci (CL)" methodology discussed in [11] and earlier in these proceedings [12] uses the eigenvalues of the transfer matrix.

Singular value loop-shaping capabilities of these various procedures are examined in [1] and [2]. They turn out to be indirect and not generally reliable. Elaboration of this judgment will be left to the references, however, and we will not study these methods further here. Rather, we will explore an alternate multivariable design technique -- the Linear-Quadratic-Gaussian (LQG) methodology [13, 14]. Although this procedure was not originally designed to perform direct singular value loop-shaping, it currently appears to be the only systematic procedure available which offers this capability.

#### The LQG Procedure

As is well-known, the LQG methodology was conceived to perform control system synthesis on the basis of time domain optimization. The method uses the following state space description of the plant, G:

$$\dot{x} = Ax + Bu + \xi \quad (2)$$

$$y = Cx + \eta \quad (3)$$

where  $x$  is an  $n$ -dimensional state vector,  $u$  and  $y$  are  $m$ -dimensional control inputs and  $r$ -dimensional measurements, respectively, and  $\xi$  and  $\eta$  are white noise processes. The  $A$ ,  $B$ , and  $C$  matrices satisfy

$$G(s) = C\Phi(s)B \quad (4)$$

$$d(s) = C\Phi(s)\xi(s) \quad (5)$$

with

$$\Phi(s) = (sI - A)^{-1}. \quad (6)$$

The feedback design task is then posed as an optimization problem to find a control law

$$u(t) = f[y(\tau); \tau < t] \quad (7)$$

to minimize the following performance criterion\*

$$J = E \left( \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (z^T z + \rho u^T u) dt \right) \quad (8)$$

with

$$z = Hx \quad (9)$$

The vector,  $z$ , is an  $m$ -dimensional response weighted in (8),  $\rho$  is a scalar parameter, and  $E(\cdot)$  denotes the usual mathematical expectation.

The solution of this optimization problem is given by [13]

$$u^*(t) = -K_C \hat{x}(t), \quad (10)$$

where  $K_C$  is a full-state linear-quadratic regulator (LQR) gain defined by the familiar Riccati equation

$$0 = PA + A^T P + H^T H - PB \frac{1}{\rho} B^T P \quad (11)$$

$$K_C = \frac{1}{\rho} B^T P \quad (12)$$

\*This criterion represents no loss of generality over the usual form [13] with weighting matrices  $Q=Q^T>0$  and  $R=R^T>0$ . The matrix  $R$  can be incorporated in  $B$  via  $B'=BR^{1/2}$  and  $H^T H$  is the most general form necessary for  $Q$  [15].

and the state estimate  $\hat{x}(t)$  is given by the Kalman filter (KBF);

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f[y(t) - C\hat{x}(t)] \quad (13)$$

whose gain,  $K_f$ , is defined by the filter's own Riccati equation

$$0 = A\Sigma + \Sigma A^T + \Gamma\Gamma^T - \Sigma C^T \frac{1}{\mu} C\Sigma \quad (14)$$

$$K_f = \Sigma C^T \frac{1}{\mu} \quad (15)$$

Here  $\Gamma\Gamma^T$  is the intensity matrix of the process noise,  $\xi$ , and  $\mu I$  with  $\mu$  scalar is the intensity matrix of the sensor noise,  $\eta$ .\*\*\*

It follows from (10) and (13) that the optimal LQG controller is a cascaded combination of a KBF and a full-state LQR. For our purposes, this combination can be treated as an ordinary FDLTI compensator:

$$K(s) = K_c(sI - A + BK_c + K_fC)^{-1}K_f. \quad (16)$$

This compensator has the special internal structure shown in Figure 3. In terms of our previous discussions, the functions of interest in Figure 3 are the loop transfer, return difference, and inverse return difference functions

$$GK, I_r + GK, I_r + (GK)^{-1}, \quad (17)$$

and also their counterparts

$$KG, I_m + KG, I_m + (KG)^{-1} \quad (18)$$

The first three functions measure performance and stability robustness with respect to uncertainties at the plant outputs (loop-breaking point (i) in Figure 3), and the second three measure performance and robustness with respect to uncertainties at the plant input (loop-breaking point (ii) in Figure 3). Both points are generally significant in design.

Two other loop-breaking points, (i)' and (ii)', are also shown in the figure. These are internal to the compensator and therefore have little direct significance. However, they have desirable loop transfer properties which can be related to the properties of points (i) and (ii). The properties and connections are these:

**Fact 1** The loop transfer function obtained by breaking the LQG loop at point (i)' is the full-state KBF loop transfer function  $C\Phi K_f$ .

**Fact 2** The loop transfer function obtained by breaking the LQG loop at point (i) is  $GK$ . It can be made to approach  $C\Phi K_f$  pointwise in  $s$  by designing the LQR in accordance with a "sensitivity recovery" procedure due to Kwakernaak [16].

**Fact 3** The loop transfer function obtained by breaking the LQG loop at point (ii)' is the full-state LQR loop transfer function  $K_c\Phi B$ .

**Fact 4** The loop transfer function obtained by breaking the LQG loop at point (ii) is  $KG$ . It can be made to approach  $K_c\Phi B$  pointwise in  $s$  by designing the KBF in accordance with a "robustness recovery" procedure due to Doyle and Stein [17].

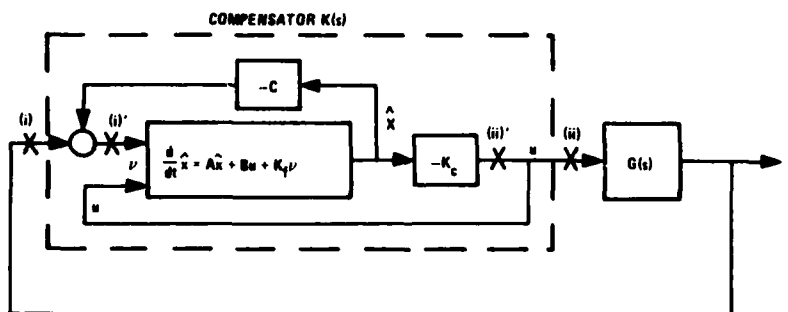


Figure 3.

\*\*\*As for the regulator, there is no loss of generality in these selections of statistics.

Facts 1 and 3 can be readily verified by explicit evaluation of the transfer functions involved. Facts 2 and 4 take more elaboration and are taken up in a later section. They also require more assumptions. Specifically,  $G(s)$  must be minimum phase with  $m > r$  for Fact 2,  $m < r$  for Fact 4, and hence,  $G(s)$  must be square for both. Also, the names "sensitivity recovery" and "robustness recovery" are overly restrictive. "Full-state loop transfer recovery" is perhaps a better name for both procedures, with the distinction that one applies to points (i), (i)' and the other to points (ii), (ii)'.

The significance of these four facts is that we can construct LQG loop transfer functions in the following two-step fashion:

Step 1: Design a full-state feedback law with desirable singular value properties. This is easy to do, as we will see shortly.

Step 2: Recover (approximate) this full-state loop transfer with a realizable LQG control law using one of the recovery procedures.

In order to apply this two-step approach, we must first decide which loop-breaking point to design for. If we select point (i), then the full state design must be done with the KBF design equations (i.e., its Riccati equation (14) - (15) and recovery with the LQR equations (11) - (12). If we select point (ii), then full-state design must be done with the LQR equations and recovery with the KBF. The mathematics of these two options are, in fact, dual. Hence, we will describe only one option (for point (ii)) in detail. Results for the other are merely summarized and then used later in an example.

#### Full-State Loop Transfer Design

The intermediate full-state design step is worthwhile because LQR and KBF loops have good classical properties which have been re-discovered over the last few years [18] - [20]. The basic result for the LQR case is that LQR loop transfer matrices

$$T(s) = K_C \Phi(s)B \quad (19)$$

satisfy the following return difference identity;

$$\begin{aligned} [I_m + T(j\omega)]^* [I_m + T(j\omega)] &= I_m + [H\Phi(j\omega)B]^* [H\Phi(j\omega)B] / \rho \\ \text{for all } 0 \leq \omega < \infty \end{aligned} \quad (20)$$

This identity can be derived directly from the LQR's Riccati equation [19]. Using the following definition of singular values ( $\sigma$ ) in terms of eigenvalues ( $\lambda$ );

$$\sigma^2[M] = \lambda[M^*M], \quad (21)$$

the return difference identity implies that

$$\begin{aligned} \sigma[I_m + T(j\omega)] &= \sqrt{\lambda[I + (H\Phi B)^* H\Phi B] / \rho} \\ &= \sqrt{1 + \lambda[(H\Phi B)^* H\Phi B] / \rho} \\ &= \sqrt{1 + \sigma^2[H\Phi(j\omega)B] / \rho} \end{aligned} \quad (22)$$

This expression applies to all singular values of  $T(s)$  and, hence, specifically to  $\underline{\sigma}$  and  $\bar{\sigma}$ . It governs the performance, crossover, and stability robustness properties of LQR loops.

Performance Properties -- Whenever  $\sigma[T] \gg 1$ , the following approximation of (22) shows explicitly how the parameters  $\rho$  and  $H$  influence  $T(s)$ :

$$\sigma[T(j\omega)] \approx \sigma[H\Phi(j\omega)B] / \sqrt{\rho} \quad (23)$$

We can thus choose  $\rho$  and  $H$  explicitly to satisfy the performance requirements in Figure 2 (high gains at low frequencies) and also to "balance" the multivariable loop such that  $\underline{\sigma}[T]$  and  $\bar{\sigma}[T]$  are reasonably close together. This second objective is consistent with our assumption in Section 2 that the transfer function  $G(s)$  has been scaled and/or transformed such that the uncertainty bound,  $\ell_m(\omega)$ , applies more or less uniformly in all directions. We also note that it may be necessary to append additional dynamics in order to meet the performance requirements. To achieve zero steady state errors, for example,  $\sigma[H\Phi B]$  must tend to infinity as  $\omega \rightarrow 0$ . This requires  $m$  free integrations in the plant which must be added if none exist to start with.

Crossover Properties -- Under mild assumption on  $A$ ,  $B$ ,  $C$  and  $H$ , the LQG regulator is nominally stable [13]. It also follows from (22) that its return difference always exceeds unity: i.e.,

$$\sigma[I_m + T(j\omega)] \geq 1 \quad \text{for all } 0 \leq \omega < \infty \quad (24)$$

This further implies [21] that

$$\sigma[I_m + T^{-1}(j\omega)] \geq 1/2 \text{ for all } 0 \leq \omega < \infty \quad (25)$$

Hence, LQR loops automatically exhibit well-behaved crossovers.

**Robustness Properties** -- It follows from (25) that LQR loops are guaranteed to remain stable for all unstructured uncertainties (reflected to the input) which satisfy  $\ell_m(\omega) < 0.5$ . Without further knowledge of the types of uncertainties present in the plant, this bound is the greatest robustness guarantee which can be ascribed to the regulator.\* While it is reassuring to have a guarantee at all, the  $\ell_m < 0.5$  bound is clearly inadequate for the uncertainty tolerance requirements of Figure 2 with realistic  $\ell_m(\omega)$ 's. In order to satisfy these requirements in LQR designs, therefore, it becomes necessary to directly manipulate the high-frequency behavior of  $T(s)$ . This behavior can be derived from known asymptotic properties of the regulator as the scalar  $\rho$  tends to zero [15, 16, 22, 23, 24]. The result needed here is that under minimum phase assumptions on  $H\Phi B$ , the LQR gains  $K_C$  behave asymptotically as [16]

$$\sqrt{\rho} K_C \Rightarrow WH \quad (26)$$

where  $W$  is an orthonormal matrix. The LQR loop transfer function,  $T(s)$ , evaluated at high frequencies,  $s = jc/\sqrt{\rho}$  with  $c$  constant, is then given by\*\*

$$T(jc/\sqrt{\rho}) = \sqrt{\rho} K_C (jcI - \sqrt{\rho} A)^{-1} B \Rightarrow WHB/jc. \quad (27)$$

Since crossovers occur at  $\sigma_i[T] = 1$ , this means that the maximum (asymptotic) crossover frequency of the loop is

$$\omega_{cmax} = \sigma[HB]/\sqrt{\rho}. \quad (28)$$

As shown in Figure 2, this frequency cannot fall much beyond  $\omega_g$ , where unstructured uncertainty magnitudes,  $\ell_m$ , approach unity. Hence, our choice of  $H$  and  $\rho$  to achieve the performance objectives via (23) are constrained by the stability robustness requirement via (28).

Note also from (27) that the asymptotic loop transfer function in the vicinity of crossover is proportional to  $1/\omega$  (-1 slope on log-log plots). This is a relatively slow attenuation rate which is the price the regulator pays for its excellent return difference properties (24). If  $\ell_m^{-1}(\omega)$  attenuates faster than this rate, further reduction of  $\omega_c$  may be required. It is also true, of course, that no physical system can actually maintain a  $1/\omega$  characteristic indefinitely [7]. This is not a concern here since  $T(s)$  is a nominal (design) function only and will later be approximated by one of the full-state loop transfer recovery procedures.

#### Full-State Loop Transfer Recovery

As described earlier, the full-state loop transfer function designed above for point (ii)' can be recovered at point (ii) by a modified KBF design procedure. The required assumptions are that  $r > m$  and that  $C\Phi B$  is minimum phase. The procedure then consists of two steps:

- (i) Append additional dummy columns to  $B$  and zero row to  $K_C$  to make  $C\Phi B$  and  $K_C\Phi B$  square ( $r \times r$ ).  $C\Phi B$  must remain minimum phase.
- (ii) Design the KBF with modified noise intensity matrices,

$$E(\xi\xi^T) = [\Gamma\Gamma^T + qBB^T] \delta(t - \tau)$$

where  $\Gamma\Gamma^T$  is the nominal noise intensity matrix obtained from stochastic models of the plant and  $q$  is a scalar parameter which will be allowed to take on a sequence of increasingly larger values.

Under these conditions, it is known that the filter gains  $K_f$  have the following asymptotic behavior as  $q \Rightarrow \infty$  [17]:

$$K_f/\sqrt{q} \Rightarrow BW \quad (29)$$

Here  $W$  is another orthonormal matrix, as in (26). When this  $K_f$  is used in the loop transfer expression for point (ii), we get point-wise loop transfer recovery as  $q \Rightarrow \infty$ , i.e.,

$$K(s)G(s) = K_C[\Phi^{-1} + BK_C + K_fC]^{-1}K_fC\Phi B \quad (30)$$

$$= K_C[\hat{\Phi} - \hat{\Phi}K_f(I_r + C\hat{\Phi}K_f)^{-1}C\hat{\Phi}]K_fC\Phi B \quad (31)$$

$$= K_C\hat{\Phi}K_f(I_r + C\hat{\Phi}K_f)^{-1}C\Phi B \quad (32)$$

\*The  $\ell_m < 0.5$  bound turns out to be tight for pure gain changes; i.e.,  $0.5 \Rightarrow 6$  db, which is identical to regulator's celebrated guaranteed gain margin [20]. The bound is conservative if the uncertainties are known to be pure phase changes, i.e.,  $0.5 \Rightarrow 30$  deg, which is less than the known  $\pm 60$  deg guarantee [20].

\*\*This specific limiting process is appropriate for the so-called generic case [22] with full rank  $HB$ . More general versions of (28) with rank  $[HB] < m$  are derived in Reference [25].

$$\Rightarrow K_C \hat{\Phi} B (C \hat{\Phi} B)^{-1} C \Phi B \quad (33)$$

$$= K_C \Phi B (I_r + K_C \Phi B)^{-1} [C \Phi B (I_r + K_C \Phi B)^{-1}]^{-1} C \Phi B \quad (34)$$

$$= \{K_C \Phi B (C \Phi B)^{-1}\} C \Phi B \quad (35)$$

$$= K_C \Phi B \quad (36)$$

In this series of expressions,  $\hat{\Phi}$  was used to represent the matrix  $(sI_n - A + BK_C)^{-1}$ , equation (29) was used to get from (32) to (33), and the identity  $\Phi B = \Phi B (I + K_C \Phi B)^{-1}$  was used to get from there to (34). The final step shows explicitly that the asymptotic compensator  $K(s)$  (the bracketed term in (35)) inverts the nominal plant (from the left) and substitutes the desired LQR dynamics. The need for minimum phase is thus clear, and it is also evident that the entire recovery procedure is only appropriate as long as the target LQR dynamics satisfy Figure 2's constraints (i.e., as long as we do not attempt inversion in frequency ranges where uncertainties do not permit it). Closer inspection of (30) - (36) further shows that there is no dependence on LQR or KBF optimality of the gains  $K_C$  or  $K_f$ . The procedure requires only that  $K_f$  be stabilizing and have the asymptotic characteristic (29). Thus, more general state feedback laws can be recovered (e.g., pole placement), and more general filters can be used for the process (e.g., observers).\*

#### Dual Results

As indicated earlier, the full-state design and recovery procedures for points (i) and (ii) in Figure 3 are mathematical duals. This means that the design equations which were developed for point (ii) above can be transformed to point (i) by symbol substitutions and transpositions. The major results and corresponding equations are the following.

#### Full-State KBF Design:

$$T(s) = C \Phi(s) K_f \quad (19)'$$

$$(I_r + T)(I_r + T)^* = I_r + (C \Phi \Gamma) (C \Phi \Gamma)^* / \mu; \text{ for } s = j\omega, 0 \leq \omega < \infty \quad (20)'$$

$$\sigma \{I_r + T\} = \sqrt{1 + \sigma^2 \{C \Phi \Gamma\} / \mu} \geq 1 \quad (22)' \quad (24)'$$

$$\sigma \{T\} \approx \sigma \{C \Phi \Gamma\} / \sqrt{\mu}; \text{ for } \sigma \{T\} \gg 1 \quad (23)'$$

$$\sqrt{\mu} K_f \Rightarrow W \Gamma \text{ as } \mu \Rightarrow 0 \quad (26)'$$

$$\omega_{cmax} = \sigma \{C \Gamma\} / \sqrt{\mu} \quad (28)'$$

Note that the "weight" selection process to achieve good singular values now focuses on the matrix  $\Gamma$  and the scalar  $\mu$  instead of  $H$  and  $\rho$ .

#### Full-State Loop Transfer Recovery Steps:

- (i) Assume  $m \geq r$ . Append dummy rows to  $C$  and  $K_f$  such that  $C \Phi K_f$  and  $C \Phi B$  are square ( $m \times m$ ) and  $C \Phi B$  is minimum phase.

- (ii) Design the LQR with modified weighting matrices:

$$Q = H^T H + q C^T C$$

Then, as  $q \Rightarrow \infty$ , we get

$$K_C / \sqrt{q} \Rightarrow W C \quad (29)'$$

and

$$G(s)K(s) \Rightarrow C \Phi B \{(C \Phi B)^{-1} C \Phi K_f\} \quad (35)'$$

$$= C \Phi K_f \quad (36)'$$

Note that asymptotic inversion now occurs from the right and that it is again appropriate only as long as the target KBF dynamics satisfy Figure 2's constraints.

#### 4.0 An Example

The full-state design and full-state recovery processes described above together provide a systematic way to shape the singular value plots of multivariable feedback loops. This is illustrated by the following abstracted longitudinal control design example for a CH-47 tandem rotor helicopter. Our objective is to control two measured outputs -- vertical velocity and pitch attitude -- by manipulating collective and differential collective rotor thrust commands. A nominal model for the dynamics relating these variables at 40 knot airspeed is [26]

\*Still more generally, the modified KBF procedure will actually recover full-state feedback loop transfer functions at any point,  $u'$ , in the system for which  $C \Phi B'$  is minimum phase [17].

$$\frac{d}{dt} x = \begin{bmatrix} -.02 & .005 & 2.4 & -32. \\ -.14 & -.44 & -1.3 & -30. \\ 0 & .018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} .14 & -.12 \\ .36 & -8.6 \\ .35 & .009 \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{bmatrix} x$$

where the state vector  $x$  consists of forward velocity, vertical velocity, pitch rate, and pitch attitude. The measurements are vertical velocity in ft/sec and attitude in degrees. Major unstructured uncertainties associated with this model are due to neglected rotor dynamics and unmodeled rate limit nonlinearities. These are discussed at greater length in [27]. For our present purposes, it suffices to note that they are uniform in both control channels and that  $\ell_m(\omega) > 1$  for all  $\omega > 10$  rad/sec. Hence, the controller bandwidth should be constrained as in Figure 2 to  $\omega_{\text{cmax}} \leq 10$ .

Since our objective is to control two measured outputs at point (i), the LQG design iterations utilize the dual equations (19)-(36). They begin with a full state KBF design whose noise intensity matrices,  $E(F\delta T) = \Gamma\Gamma^T\delta(t-\tau)$  and  $E(\eta\eta^T) = \mu I\delta(t-\tau)$ , are selected to meet performance objectives at low frequencies; i.e.,

$$\underline{\sigma} [T] \leq \underline{\sigma} [C \Phi \Gamma] / \sqrt{\mu} \geq \text{ps}(\omega), \quad (37)$$

while satisfying stability robustness constraints at high frequencies;

$$\omega_{\text{cmax}} = \bar{\sigma} [C \Gamma] / \sqrt{\mu} < 10 \text{ r/sec} \quad (38)$$

For the choice  $\Gamma = B$ , equation (38) constrains  $\mu$  to be greater than or equal to unity.\* The resulting KBF loop transfer for  $\mu = 1$  is shown in Figure 4. For purposes of illustration, this function will be considered to have adequate high gain properties for performance with low gains beyond  $\omega = 10$  for stability robustness.\*\* It then remains to recover this function by means of the full-state recovery procedure for point (i). This calls for LQR design with  $Q = H^TH + q C^TC$  and  $R = \rho I$ . Letting  $H = 0$ ,  $\rho = 1$ , the resulting LQG transfer functions for several values of  $q$  are also shown in Figure 4. They clearly display the pointwise convergence properties of the procedure.

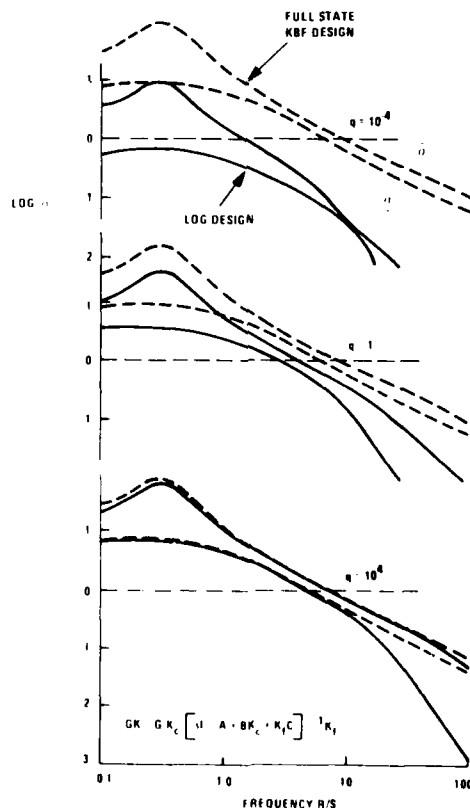


Figure 4.

\*If  $C\Gamma$  (or  $H B$ ) is singular, equations (38) or (28) are still valid in the non-zero directions.

\*\*The function should not be considered final, of course. Better balance between  $\bar{\sigma}$  and  $\underline{\sigma}$  and greater gain at low frequencies via appended integrators would be desirable in a serious design.

## 5.0 Conclusions

We have shown how the modern time domain LQG methodology can be used to design multivariable feedback controllers which satisfy practical design requirements expressed as singular value conditions in the frequency domain. The design process was broken into two basic steps. First, a full-state LQR (or KBF) feedback law is designed whose singular value properties meet design objectives -- high gains where needed for performance, low gains where needed for uncertainty tolerance, and well-behaved crossovers in between. Quadratic weights for this step are determined explicitly by known relationships between the weights and resulting full-state controller's singular value properties. In the second step, this full-state loop transfer matrix is then approximated with a realizable LQG compensator using a loop transfer recovery procedure. This procedure involves the traditional KBF design (or LQR design, respectively), but with modified noise statistics. The modifications cause the LQG loop transfer matrix to approach the full-state matrix arbitrarily closely at fixed  $s$ . The overall design process was illustrated with a simple example which shows it to be a direct and convenient way to manipulate multivariable singular value properties.

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## LQG MULTIVARIABLE DESIGN TOOLS

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## Summary

This paper describes the basic design algorithms needed for frequency domain oriented Linear-Quadratic-Gaussian feedback design and introduces an experimental interactive computer aided design package through which these algorithms can be effectively accessed. The algorithms and design package are illustrated with several flight control design examples for highly maneuverable aircraft.

## 1.0 INTRODUCTION

In an earlier paper in these proceedings [1], a procedure was discussed by which the Linear-Quadratic-Gaussian (LQG) synthesis methodology [2] can be used for frequency-domain oriented multivariable control system design. This procedure is based on the interpretation of various design requirements as specifications on the singular value plots of the design's loop transfer function matrices. The resulting desired loop shapes are then achieved with a two-step LQG process:

Step 1: Full-state design with quadratic weights or noise statistics selected to satisfy the singular value specifications.

Step 2: Full-state loop transfer recovery with modified filter or regulator design parameters.

Except for the most elementary design problems, these steps cannot be carried out effectively with manual calculations. Computerized design assistance is essential, either in the traditional batch-processing mode or, preferably, in an automated interactive design environment.

This paper discusses some of the basic elements of a computerized LQG design process. It begins with a brief review of the LQG methodology. The core algorithms needed for LQG synthesis are then discussed, and a prototype command structure is introduced for their interactive computer implementation. This command structure is currently hosted on a Honeywell Multics system and is being used for various interactive control design experiments. The structure is illustrated with several aircraft flight control design examples.

## 2.0 REVIEW OF THE LQG LOOP SHAPING PROCESS

As in [1] and [3], we will deal with the generic feedback loop shown in Figure 1. It consists of the interconnected plant ( $G$ ) and compensator ( $K$ ) forced by commands ( $r$ ), measurement noise ( $n$ ), and by a set of disturbances ( $d$ ) which are assumed to be reflected to the measured output ( $y$ ). The dashed precompensator ( $P$ ) is an optional element used to achieve deliberate command shaping or to represent a non-unity feedback system in equivalent unity feedback form. All signals are taken to be multivariable, in general, and both nominal mathematical models of  $G$  and  $K$  are finite dimensional linear time-invariant systems with transfer function matrices  $G(s)$  and  $K(s)$ .

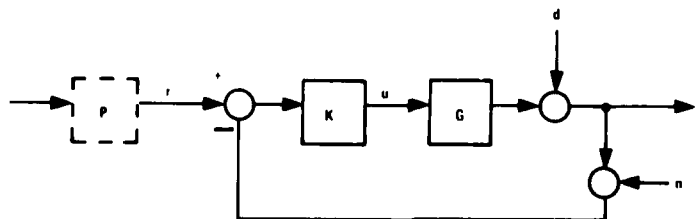


Figure 1. Standard Feedback Configuration

Basic Design Requirements

As discussed in [3], the feedback system in Figure 1 has good performance properties whenever the inverse return difference operator  $(I+GK)^{-1}$  is "small". For scalar transfer functions, the appropriate notion of smallness can be captured by absolute values; i.e., we typically require that  $1/|1+g(j\omega)k(j\omega)|$  be small compared to unity

in a frequency range where commands and/or disturbances are large. This is typically the low frequency range, say  $\omega < \omega_0$ . In the multivariable case, an analogous notion of smallness has been defined based on singular values [3]. We require that the maximum singular value of  $(I+GK)^{-1}$ , denoted by  $\sigma[(I+GK(j\omega))^{-1}]$ , be small compared to unity over the designated frequency range. The converse of this requirement is that the return difference operator  $I+GK$  be "large", i.e.,

$$\sigma[I+GK(j\omega)] > \psi(\omega) \gg 1$$

for all  $\omega \geq \omega_0$ , (1)

where  $\psi(\omega)$  is a scalar function typically available as an explicit frequency domain performance specification. With  $\psi(\omega)$  sufficiently large, this expression simply states that the loop gains  $\sigma[GK(j\omega)]$  must themselves be large. Evidently, good feedback design boils down to achieving high loop gains in the necessary frequency range.

Of course, this latter statement expresses only half of the feedback design problem. In real physical feedback systems, loop gains can never be made arbitrarily high over arbitrarily wide frequency ranges. Rather, they are constrained by our desire to suppress sensor noise responses and also by the requirement to remain stable in the presence of model uncertainties.

The second source of limitations is commonly the more severe. The feedback system is required to operate successfully in the presence of an infinite variety of differences between the actual plant and its design model  $G(s)$ . A very useful way to express this variety of possible differences is in the so-called unstructured multiplicative form:

$$G'(s) = [I+L(s)] G(s) \quad (2)$$

$$\text{with } \overline{\sigma}[L(j\omega)] \leq \ell_m(\omega)$$

Here the actual plant  $G'$  is represented by its nominal model,  $G(s)$ , modified by a multiplicative perturbation,  $L(s)$ , whose characteristics are known only by the magnitude bound  $\ell_m(\omega)$ . Typically, this bound is small ( $\ell_m \ll 1$ ) at low frequencies, but grows with increasing frequencies and eventually exceeds unity at frequencies  $\omega \geq \omega_g$ .

As shown in [3], a nominally stable feedback system will remain stable in the face of unstructured uncertainties of the type (2) if and only if

$$\sigma[I+(GK)^{-1}] > \ell_m(\omega) \quad (3)$$

$$\text{for all } 0 \leq \omega < \infty$$

This equation is a multivariable generalization of the familiar single loop requirement that loop gains be small whenever uncertainties are large. In fact, whenever  $\ell_m(\omega) \gg 1$ , we get the following constraint on  $GK$ :

$$\overline{\sigma}[GK(j\omega)] < 1/\ell_m(\omega) \ll 1 \quad (4)$$

$$\text{for all } \omega \text{ such that } \ell_m(\omega) \ll 1$$

Based upon (1) - (4), it follows that the fundamental feedback design problem becomes one of finding a compensator  $K(s)$  which shapes the loop transfer matrix  $GK(s)$  in such a way that

- a) the loop gains  $\sigma[GK]$  are high at low frequencies to meet performance requirements (1),
- b) the loop gains  $\sigma[GK]$  are low at high frequencies to meet the stability robustness requirements (4), and
- c) the transitions, or "crossovers", between these two regions are "well-behaved". This means that they are stable (i.e.,  $\det(I+GK)$  satisfies the multivariable Nyquist Criterion [4]) and that  $I+GK$  and  $I+(GK)^{-1}$  are not unduly small.

#### LQG Loop-Shaping

Although the Linear-Quadratic-Gaussian methodology was developed purely from a time domain optimization perspective, we saw in [1] that it proved to be remarkably effective as a tool for the frequency-domain loop shaping task outlined above. In order to briefly review the results of [1], we will deal with the standard LQG controller configuration shown in Figure 2. It consists of the well-known cascaded combination of a Kalman-Bucy filter (KBF) with a full-state linear-quadratic regulator (LQR). Beyond this special internal structure, the combination comprises an ordinary linear finite dimensional compensator with transfer matrix

$$K(s) = K_C(sI-A+BK_C+K_F C)^{-1}K_F \quad (5)$$

where the standard matrix triple  $(A, B, C)$  has been used to represent a state space realization of the plant, and the symbols  $K_C$  and  $K_F$  are used to represent the LQR and KBF gain matrices, respectively, as defined by their individual algebraic Riccati equations [1]:

$$\begin{aligned} K_C &= R^{-1}B^T P \\ 0 &= PA + A^T P + Q - PB^T R^{-1} B P \end{aligned} \quad (6)$$

$$\begin{aligned} K_f &= \Sigma C^T N^{-1} \\ 0 &= A \Sigma + \Sigma A^T + M - \Sigma C^T N^{-1} C \Sigma \end{aligned} \quad (7)$$

The loop transfer matrices of the LQG controller are then either  $G(s)K(s)$ , for the loop broken at point (i) in Figure 2, or  $K(s)G(s)$ , for the loop broken at point (ii). In general, these functions have no special properties except nominal closed loop stability. However, if the plant  $G(s)$  is square and minimum phase, and if we design the LQR with the following weighting matrices;

$$\begin{aligned} Q &= qC^T C \quad (\text{scalar } q \Rightarrow \infty) \\ R &= \rho I \quad (\text{scalar } \rho > 0 \text{ and fixed}) \end{aligned} \quad (8)$$

then the loop transfer function  $GK(s)$  will approach a certain full-state loop transfer function arbitrarily closely as  $q \Rightarrow 0$ ; namely

$$GK(s) \Rightarrow C(sI-A)^{-1}K_f = T(s) \quad (9)$$

The function  $T(s)$  turns out to be the loop transfer function of the Kalman-Bucy filter's error dynamics and can be readily adjusted to exhibit all three of the loop-shaping design objectives discussed above. In particular, if we let the filter's noise statistics be

$$\begin{aligned} M &= \Gamma \Gamma^T \\ N &= \mu I, \quad (\text{scalar } \mu > 0) \end{aligned} \quad (10)$$

then  $T(s)$  in (9) satisfies

$$a) \quad \sigma[T(j\omega)] \sim \sigma[C(j\omega I-A)^{-1}\Gamma]/\sqrt{\mu} \quad \text{for all } \omega \text{ such that } \sigma[T] \gg 1 \quad (11)$$

$$b) \quad \sigma[T(j\omega)] \Rightarrow \sigma[C\Gamma]/\omega\sqrt{\mu} \quad \text{as } \omega \Rightarrow \infty, \mu \Rightarrow 0 \text{ with } \omega\sqrt{\mu} \text{ constant} \quad (12)^*$$

$$c) \quad \sigma[I+T(j\omega)] \geq 1 \quad (13)$$

$$\begin{aligned} &\text{and} \\ &\sigma[I+T^{-1}(j\omega)] \geq 1/2 \\ &\text{for all } 0 \leq \omega < \infty \end{aligned} \quad (14)$$

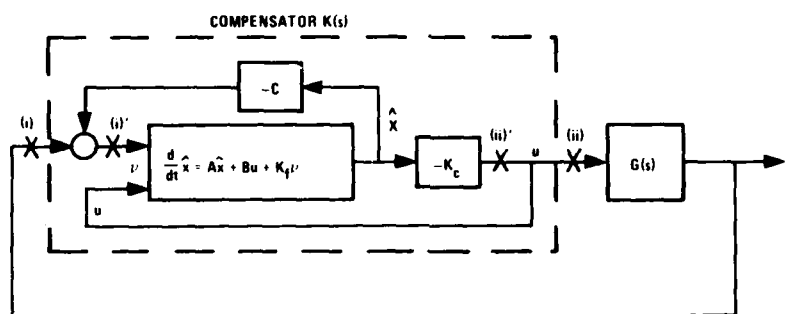


Figure 2. LQG Controller

Note that property a) provides a direct way to satisfy our high gain objectives at low frequencies. The gains are simply given by the noise-to-output transfer function  $C(sI-A)^{-1}\Gamma/\sqrt{\mu}$ , with  $\Gamma$  and  $\mu$  at our discretion. Similarly, property b) provides a way to meet the stability robustness requirement at high frequencies. It shows that the loop gains become small at an asymptotic rate proportional to  $1/\omega$  and achieve their last crossover at the frequency

$$\omega_{cmax} = \sigma[C\Gamma]/\sqrt{\mu}. \quad (15)$$

Finally, property c) shows that all crossovers will be "well-behaved". The return difference always exceeds unity, the inverse return difference always exceeds 0.5, and of course, the nominal loop is guaranteed to be stable.

Based on the above summary of frequency domain properties of LQG controllers, the following simple loop-shaping procedure suggests itself:

Step 1. Design a KBF with  $\Gamma$  and  $\mu$  selected such that the loop transfer function  $T(s)$  meets performance and stability robustness requirements.

\*This equation applies to the "generic" case with  $C\Gamma$  full rank [1].

Step 2. Design a sequence of LQR's with weights selected according to (6) and  $q$  allowed to take on consecutively larger values. Then select an element of the resulting sequence of transfer functions

$$GK(s) \Rightarrow T(s)$$

which adequately "recovers" the desired function over the frequency range of interest. All design objectives including nominal stability are then assured.

As discussed in [1], a dual version of this process is also available for shaping the loop transfer function  $KG(s)$  at point (ii) in Figure 2. For this point, however, the role of the filter and controller are reversed. We begin by designing an LQR whose loop transfer  $K_C(sI-A)^{-1}B$  has good frequency domain properties and then design a sequence of KBFs which serve to "recover" this function. Equations for this alternate procedure are mathematical "duals" of the results given above. Both procedures are illustrated in the design examples of Section 4.

### 3.0 COMPUTATIONAL TOOLS FOR LQG DESIGN

As noted earlier, the design steps outlined above can be carried out manually for only the most elementary problems. Even then, the calculations are likely to be tedious. Hence, serious use of the LQG method in the design environment of industry and in the teaching environment of academe requires computerized assistance. The basic algorithms needed for this purpose and a prototype interactive command structure for easy user access to these algorithms are discussed below.

#### Basic Algorithms

It follows from Section 2 that the core computing tasks of our frequency-domain-oriented LQG synthesis process include the following operations:

1. Riccati solutions for gains  $K_C$  and  $K_f$
2. Frequency response evaluations for  $G(j\omega)$ ,  $K(j\omega)$ ,  $GK(j\omega)$ , etc.
3. Singular value decompositions

FORTTRAN subroutines for these operations are available on many scientific computing systems. Riccati equation solvers, for example, exist in several software packages [5-7]. They generally compute solutions for equation (6) and for its mathematical dual (7) by means of the so-called Potter algorithm [8]. This algorithm expresses the solution matrix,  $P$ , in terms of a full eigenvalue/eigenvector decomposition of the  $2n$ -dimensional Hamiltonian system associated with the LQ-optimization problem [9]. That is, let

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} = W \begin{bmatrix} \Lambda^- & 0 \\ 0 & \Lambda^+ \end{bmatrix} W^{-1}$$

where  $H$  is the Hamiltonian system matrix,  $\Lambda^- = \text{diag}(\lambda_i^-)$  and  $\Lambda^+ = \text{diag}(\lambda_i^+)$  are its stable and unstable (mirror image) eigenvalues respectively, and where  $W$  denotes the corresponding matrix of eigenvectors. Then  $P$  is defined in terms of a  $2 \times 2$  block partition of  $W$  as

$$P = W_{21} (W_{11})^{-1} \quad (17)$$

An alternate algorithm is available due to Kleinman [10] which utilizes Newton iterations to solve the Riccati equation. It requires a stabilizing initial guess for  $K$  or  $P$  and then iterates as follows:

- (i) Let  $K = R^{-1}B^T P_0$
  - (ii) Solve  $0 = P_1(A-BK) + (A-BK)^T P_1 + Q + KRK^T$  for  $P_1$
  - (iii) If  $\|P_1 - P_0\| \geq \epsilon$  return to (i) with  $P_0 = P_1$
- Otherwise terminate with  $P = P_1$

Still more recently, an algorithm has been introduced by Laub [11] which generalizes Potter's method by using a real Schur form in place of the full eigenvalue/vector decomposition to isolate the stable subspace of the Hamiltonian system. In the real Schur form, the matrix analogous to  $\Lambda^-$  in (16) is upper triangular with real eigenvalues on the diagonal and selected nonzero entries on the first subdiagonal corresponding to pairs of complex conjugate roots. In addition, the transformation  $W$  is real and orthonormal (i.e.,  $W^{-1} = W^T$ ). This offers substantial numerical advantages and makes Laub's algorithm probably the most convenient and numerically robust Riccati solver available at this time.

Transfer function and frequency response evaluations are also standard operations in many computing systems. Given a state space tripe  $A, B, C$  (or more generally, a quadruple  $A, B, C, D$ ), its transfer function at fixed frequency, say  $s = j\omega_i$ , is defined as

$$G(j\omega_i) = C(j\omega_i I - A)^{-1} B + D \quad (19)$$

This expression is usually evaluated in two steps. First, the linear equation system

$$(j\omega_i I - A) X = B \quad (20)$$

is solved for  $X$  via Gaussian elimination (LU factorization followed by solution of a triangular system [12]) and then  $G$  is evaluated as

$$G = CX + D \quad (21)$$

These steps require approximately  $(n^3/3 + mn^2 + rmn)$  multiply-add operation [13], where  $n$  is the state space dimension of the system (i.e.,  $n = \dim A$ ),  $m$  is the number of inputs, and  $r$  is the number of outputs. Note that this operation count is proportional to  $n^3$  and that it must be repeated for each frequency point of interest. Hence, if the state space dimension is high and if the number of frequency samples is large, frequency response evaluation can consume considerable computing time. These time demands can be reduced, however, by utilizing transformed state space representations. For example, a full eigenvalue/vector decomposition of matrix  $A$  can be used to convert the system  $A, B, C, D$  into its diagonal equivalent

$$\text{diag}(\lambda_i), W^{-1}B, CW, D \quad (22)$$

While this involves a one-time overhead of  $O(n^3)$  operation, all subsequent evaluations of (20)-(21) require only  $(mn + rmn)$  operations. Hence, the total computing time can be reduced considerably. Unfortunately, complete eigenvalue/vector decompositions are often plagued with numerical ill-conditioning, and hence, the form (20) is not always reliably available. An alternate transformation, which avoids this difficulty and still achieves significant computer savings is the Hessenberg form. This form is upper triangular with nonzero entries on the first subdiagonal. Like the Schur form, it is achieved with real orthogonal transformations using a one-time overhead of  $5n^3/3$  operations. Its structure remains intact when we add  $j\omega_i I$ , so (20)-(21) can be solved for each subsequent frequency point with approximately  $(mn/2 + rmn)$  operations [13]. Hence, whenever  $n$  is large compared with  $m$  and full eigenvalue/vector decompositions are unreliable, the Hessenberg form provides an effective alternative transformation procedure.

The third basic computing task for LQG design involves singular value decompositions, or SVDs. These are also available as standard routines on many computing systems. They convert the rxm matrix  $G(j\omega)$  into the following factors:

$$G(j\omega) = U \Sigma V^H \quad (23)$$

where  $U$  and  $V$  are  $(rxr)$  and  $(mxm)$  unitary matrices (i.e.,  $U^H U = I$ ,  $V^H V = I$ ), respectively,  $( )^H$  denotes complex conjugate transpose, and  $\Sigma$  is an rxm matrix with entries only along the main diagonal. These entries are

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_p \geq 0 \quad (24)$$

with  $p = \min(r, m)$ . The two extreme values,  $\sigma_1$  and  $\sigma_p$ , correspond to the maximum and minimum singular values,  $\bar{\sigma}$  and  $\underline{\sigma}$ . Their significance with respect to LQG design has already been discussed.

The computation of SVD's relies on a sequence of orthogonal equivalence transformations whose details are described in [12]. This same reference also provides FORTRAN listings of an SVD subroutine. SVD's consume approximately  $6r^2m$  multiply-add operations. While this is quite large compared with other equal-size matrix operations (e.g., inversion, eigenvalue/vector decomposition), the dimensions  $r$  and  $m$  are usually modest and other calculation costs dominate LQG design.

In addition to these three basic computing algorithms needed for LQG synthesis, an effective computerized design environment must also provide software facilities for more routine matrix manipulations (eg. sums, products, inverses, etc.), for input/output, data plotting, and for other data management functions. An example of these essential support capabilities is given later in this section.

#### Interactive Environments

In traditional batch-processing computer systems, basic computing algorithms such as the ones described above are accessed by the user as subroutines. They are assembled into a desired fixed sequence of operations by a user-written main program which also handles routine manipulations, input/output, data management, etc. This program is submitted to the computing facility where it is compiled and executed. Results are then returned to the user for evaluations, program modifications, and re-submittal. While this process is a substantial improvement over manual computations, it is still slow, prone to error, and frustrating for the designer. As a result, considerable research effort and hardware/software development has been devoted to its improvement.

One of the results of these R&D efforts is the interactive approach to computer-aided design. In this approach, the designer interacts with the computer directly, flexibly, and in apparent real time. He is usually provided with the following facilities:

- a graphics terminal
- a software library of design programs
- a high-level command language
- editing and macro command generation software.

The terminal provides real time computer access and control with effective graphic communication of design status, results and tradeoffs. It is usually augmented with hardcopy equipment and off-line printing facilities to provide permanent records of the communication. The subroutine library contains all design algorithms needed for a specified class of design problems (eg. Riccati solvers, SVD's, etc. for LQG design). The command language provide a simple means to invoke these high-level routines in any user-desired sequence without concerns about input/output, data handling, internal data structures and numerics. A command language should be contrasted with alternative "question and answer" approaches to computer interaction. In the latter, the computer program guides the user through a pre-programmed tree of fixed design sequences with a controlled question and answer dialogue. This approach has been shown to lack the flexibility required for a wide spectrum of users [14]. Finally, the editing and macro capability provides a way to define, save and execute fixed sequences of user-specified commands. In effect, a user can create higher-level commands suitable for his own design task. These enhance the power of the basic command language. Macro facilities have been found to be a key element in the success of interactive design [14].

#### An Illustrative Interactive System

It is not our intent here to discuss the various issues of interactive computer-aided design in detail. This is done in [14]. Rather, the intent is to illustrate the use of interaction as it might apply to the LQG design process. This will be done by means of an example interactive program package, HONEY-X, which has been constructed on Honeywell's Multics computer system [15]. The purpose of this package has been to support multivariable control design research [16-17]. As such, the package is still in a state of flux and somewhat limited in scope. It does, however, provide an effective illustration of the power of interactive design.

HONEY-X consists of a library of standard ANSI FORTRAN design programs which are controlled and accessed through the command and file management facilities offered by Multics. Multics also provides editing and macro command generation capabilities. The package is organized into three hierarchical levels, as illustrated in Figure 3. The levels are:

System Executive--This level is an executive program written in the Multics Command Language. It is responsible for establishing and maintaining the design system environment. Upon entry with the command "ec matrix", it makes the design system library available to the user by designating subprogram locations within the Multics file system. Next, the system executive solicits a command line from the user and performs first-level parsing to determine to which of the following classes the command belongs:

1. HONEY-X system command
2. User Macro command
3. Multics command

The command is then passed to the appropriate processor for execution. The system executive continues to solicit commands until it is terminated with the quit command "q".

Program Executive--This level consists of executive programs which interface between the system executive and the requested object programs. Its primary responsibility is to direct input to and output from the file system. It must perform second-level parsing in order to determine which files should be attached. If the command line contains optional arguments, these are passed to the object program.

Object Program--This level performs the actual computations which are implied by the command line. It must check for compatibility of input data. If optional arguments have been passed by the program executive, it must parse them and determine if it has been given sufficient information to proceed. If input parameters are missing and do not have default values, they must be solicited from the user. Note that prompting for input is provided only if the required information is not present. This allows experienced users to avoid unnecessary input menus while giving unfamiliar users all the information needed. When applicable, the object program has the final responsibility of directing printed output to the user's terminal I/O device.

Programs in HONEY-X's library were selected and developed in the course of several years of multivariable design research. Programs were added until they formed a set which would allow most manipulations to be performed by a straight-forward combination of two or three commands. At that point, programs with significant overlap in function were combined. This produced two basic sets of routine -- 1) the "General Matrix Routines" which perform primitive operations on matrices, and 2) the "Special-Purpose Programs" which perform more involved functions of linear systems analysis and synthesis. The special purpose programs require the existence of specific input matrices that have been named according to the established naming conventions. Similarly, their output matrix names are chosen so as to identify the contents. Table 1 provides brief description of both types of routines.

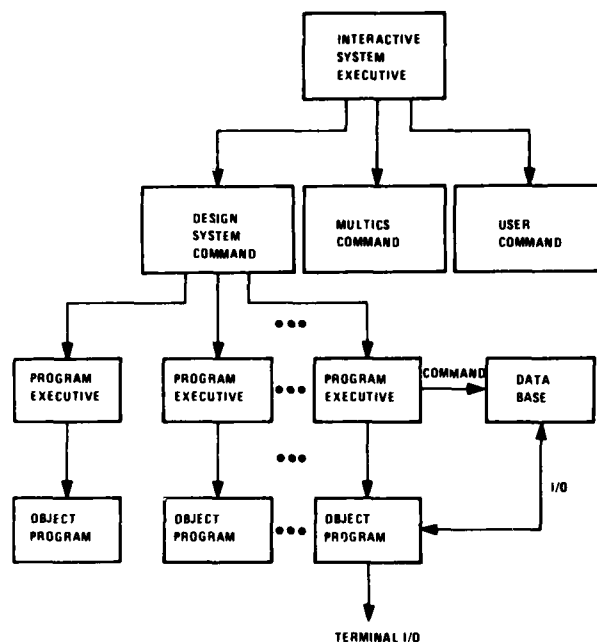


Figure 3. HONEY-X Structure

TABLE 1. HONEY-X PROGRAM LIBRARY

## I. GENERAL MATRIX ROUTINES

A. Initialization

matini (initialized, dimensions, entries)  
 matsid (scalar times identity)

B. Manipulative

mataug (augment)  
 matchg (change dimensions, entries)  
 matsam (frequency sample)  
 matsel (select rows and columns)  
 addname (Multics system routine)  
     adds new file names  
 copy (Multics system routine)  
     copies files  
 delete (Multics system routine)  
     deletes files  
 rename (Multics system routine)  
     changes file names

C. Display

matchg (print entries greater than a threshold)  
 matplt (plot)  
 matprt (print)  
 matscn (row and column dimensions; list of contained frequencies)

D. Functions

matcnj (conjugate)  
 matctp (conjugate transpose)  
 matdia (vector  $\Rightarrow$  diagonal)  
 matimg (imaginary part)  
 matinv (inverse)  
 matpol (rectangular  $\Rightarrow$  polar)  
 matrec (polar  $\Rightarrow$  rectangular)  
 matrel (real part)  
 mattrp (transpose)

E. Arithmetic Operations

matadd (addition)  
 matdiv (division)  
 matmpy (multiplication)  
 matscl (multiplication by a scalar)  
 matsub (subtraction)

- F. Coordinate Transformation  
 mateig/geneig (eigenvalue/eigenvector)  
 matnrm (normalize columns)  
 matsvd (singular value decomposition)

## II. SPECIAL-PURPOSE ROUTINES

- A. Transfer Function Generation  
 cisabd (from state-space quadruple)  
 leadlag (lead/lag compensator)  
 notch (notch/inverted notch filter)
- B. Analysis  
 abk (closed-loop eigenvalues)  
 svdsys (multivariable margins)  
 zeros (transmission zeros)
- C. Ricatti Equation Solutions  
 kalman (Kalman-Bucy filter)  
 lqgain (linear quadratic optimal control gain)

HONEY-X's command language is set up to provide commands for each of the programs in the library. These commands take the following generic form:

<command identifier> <argument list>

The <command identifier> is merely the name of one of the programs in the library (eg. matini, matplt, kalman), and the <argument list> is a string of characters which identifies data files to be used by the program and various arguments needed for its execution. A few examples serve to illustrate this command format:

```
>matmpy sys.b sys.k sys.bk
>matsub sys.a sys.bk sys.a
>cisabd sys -1,2,Q1
>matplt sys.g/Real vs. Frequency/lwlr all
```

(25)

The character > in each of these command lines is a prompting symbol displayed by the system executive on the graphics terminal to solicit a command. The command identifier is entered immediately after the prompt. For example, "matmpy" in line 1 invokes the matrix multiply program. It takes the matrix stored under file name "sys.b", multiplies by the matrix stored under filename "sys.k" and stores the results under filename "sys.bk". The string "sys.a sys.b sys.bk" constitutes a specific list of arguments for matmpy. Similarly, the second command line invokes the matrix subtraction program, matsub, which takes the matrix stored under filename "sys.a", subtracts the matrix under "sys.bk" just created by matmpy, and stores the result under "sys.a". Note that the last step deliberately overwrites the original sys.a file. Another file name supplied as a third argument would leave sys.a intact.

The third command line invokes the special purpose routine, cisabd, which evaluates a frequency response matrix according to equations (19) - (21). The argument "sys" causes this program to use A, B, C, and D matrices stored under filenames "sys.a", "sys.b", "sys.c" and "sys.d", respectively, and to store the resulting frequency response under filename "sys.g". In effect, the routine represents an operation performed on a system, not on an individual data file (eg. argument "sys" replaced with "x" would cause matrices A, B, C, D stored under files x.a, x.b, x.c, x.d to be used to produce a frequency response stored under x.g). The additional characters in the argument string of cisabd specify the desired frequency range for evaluation, in this case  $10^{-1}$  to  $10^2$  rad/sec, and the frequency increment between points, in this case  $10^{0.1}$  rad/sec or ten points per decade. If these latter three arguments are not supplied on the command line, cisabd will ask for them explicitly before execution.

The final command line in our example invokes the routine "matplt" which provides graphical plotting capability. The argument list identifies that the matrix to be plotted is stored under filename "sys.g", specifies that the plot be labeled "Real vs. Frequency", and that the x, y variables be "lw" for log frequency, and "lr", for log real, respectively. Scaling is automatic by the default option "all". Matplt will proceed to generate the desired graph on the terminal face, complete with axes and labels, and the user may then save this plot as hardcopy if its results are satisfactory.

The file name requirements and argument lists of other HONEY-X commands are defined in a similar manner and are fully documented in Reference [16].

### 4.0 DESIGN EXAMPLES

The effectiveness of HONEY-X and the LQG design process in Section 2 is best illustrated with some representative design examples. The examples selected here are taken from Reference [18]. They are concerned with the synthesis of flight control laws for a highly maneuverable aircraft. The vehicle is a scaled version of an advanced fighter presently being flight-tested on NASA Dryden's Remotely Augmented



Vehicle (RAV) facility [19]. While the present emphasis is on validating aerodynamics, control is an important aspect of these flight tests because the open loop vehicle is highly unstable. The vehicle's multiple, independently controlled surface offer the potential for high maneuverability in future experiments and an excellent test bed for multivariable design.

The present baseline set of control laws for the vehicle's pitch axis includes a single-input single-output (SISO) command-augmentation system (CAS) combined with automatic limiting of load factor and angle-of-attack. These control laws were designed using classical loop-shaping considerations [20]. The control system designs presented here represent an attempt to achieve the same basic pitch axis performance with the LQG method of Section 2 and to extend these designs to multivariable decoupled flight path control (DFPC) functions.

### The Design Model

The designs will be based on a linear model describing motion in the vertical plane. The flight condition corresponds to a Mach number of 0.9 at 25,000 ft. altitude. Linearized (small perturbation) models for this as well as other flight conditions are given in [20]. For design purposes the longitudinal dynamics are uncoupled from the lateral-directional dynamics. The state vector consists of the vehicle's basic rigid body variables

$$x = \begin{matrix} \delta V & \text{forward velocity} \\ \alpha & \text{angle-of-attack} \\ q & \text{pitch rate} \\ \theta & \text{pitch attitude} \end{matrix} \quad (26)$$

As usual, flight path angle ( $\gamma$ ) is defined as  $\gamma = \theta - \alpha$ . The control surfaces are the elevator, elevon and canard, each commanded through 70 rad/sec hydraulic actuation systems.

### Model Uncertainties

The above model neglects higher order structural modes and resonances and actuator dynamics beyond 70 rad/sec. Furthermore, because of time delays in the RAV data links, the actual transfer function differs substantially from the nominal model at high frequencies. These uncertainties define the  $\ell_m(\omega)$  bound described in Section 2. Conservatively, this bound exceeds unity for all  $\omega$  greater than about 10 rad/sec. To minimize command-following errors, disturbance responses and closed loop sensitivity to low frequency plant variations it is, of course, desirable to maximize loop gains below this frequency subject to the limitations imposed by uncertainty.

The next section considers the design of three separate control loops. The first is a minimal pitch CAS. This is followed by an angle-of-attack limiter, and finally by an advanced multivariable design for decoupled flight path control.

### A Minimal CAS

The function of the CAS is to stabilize the vehicle and allow it to be flown by commands from the ground-based RAV facility. The commanded variable is pitch rate and the control input is a slaved elevator/elevon combination. This minimizes sensing and actuation requirements. To realize this design with the LQG procedure described in Section 2, either variant of the procedure may be used, i.e., KBF design with LQR loop transfer recovery for point (i) in Figure 2 or LQR design with KBF loop transfer recovery for point (ii). This is because SISO loop transfer functions are identical at both points. We choose the latter option on historical precedent.

The first step of the design, then, is to choose LQR parameters for the weighting matrices  $Q = H^T H$  and  $R = \rho I$  (the duals of equation (10)). A first intuitive choice for  $H$  is (0010), which corresponds to weighting the commanded variable pitch rate. In light of the dual of equation (11), namely

$$\sigma[T(j\omega)] \approx \sigma[H(j\omega I - A)^{-1}B] / \sqrt{\rho} \quad (27)$$

for all  $\omega$  such that  $\sigma[T] \gg 1$ ,

the quality of this choice can be examined without actually computing the LQG regulator. Rather, we examine the singular value plots of the transfer function  $H(j\omega I - A)^{-1}B$ . This function can be readily constructed with HONEY-X commands. We simply type

```
>addname Al.a x.a
>addname Al.b x.b
>addname Al.h x.c
>cisab x -3,20.1
>matsvd x.g 1
>matplt x.g.sigma/Al: H(sI-A)B/ lwlr
```

(28)

These commands assume that we have previously defined the files Al.a, Al.b and Al.c for the aircraft dynamics and Al.h for our first choice of weights. The programs "matini" and "matchg" are available for this purpose. The three "addname" commands then

associate a dummy system name "x" with a subset of these files. That is, the {A,B,C} triple of system "x" is equivalent to the {A,B,H} triple of system "A1". The program "cisab" which expects files in the standard {A,B,C} convention can now operate on system "x" to produce the desired transfer function file x.g. Next, the program "matsvd" performs a singular value decomposition of x.g (an absolute value operation for the current SISO case), and "matplt" displays the resulting singular value file, x.g.sigma, as a function of log frequency. The resulting plot is shown in Figure 4A.

According to equation (27), the shape of Figure 4A will be the shape of the LQR loop whenever its magnitude is large. It therefore follows that our first weighting choice would produce insufficient low frequency gain for a bandwidth constraint of 10 rad/sec. An obvious way to overcome this limitation is to weight the pitch attitude response (the integral of pitch rate) instead. This makes  $H=(0001)$ . We can examine this alternative by modifying the file A1.h with "matchg" and then retyping the command list (28). On the other hand, because (28) will be used frequently, it can be saved as a macro command. For illustration, we will call this macro "lqq\_weight", and we will set the macro up to treat the string "A1" in (28) as an argument.\* The single command

>lqq\_weight A1 (29)

will then generate and display the new singular value plot shown in Figure 4B. While this alternative clearly shows improved low frequency properties, its gain at frequencies below 0.10 rad/sec is still very small. This can be remedied by introducing still more amplification into the weighted response at low frequencies. For example, we can define a new system "A2" by appending an added state to "A1" of the form

$$x_5(s) = \frac{1}{s+0.02} \theta(s). \quad (30)$$

The weighting  $\theta' = \theta + 0.14 x_5$  then gives

$$\theta'(s) = \frac{s+0.16}{s+0.02} \theta(s) \quad (31)$$

with  $H = (0 \ 0 \ 0 \ 1.0 \ 0.14)$ . The command "lqq\_weight A2" now generates Figure 4C. This weight selection exhibits very desirable loop-shape properties.

TABLE 2. MACRO lqr\_weight

```
&command_line off
addname &l.a x.a
addname &l.b x.b
addname &l.h x.c
ec cisab x &2
ec matsvd x.g 1
ec matplt x.g.sigma/&l: H{sI-A}B/ lwlr &3
dn x.a x.b x.c
delete x.g x.g.sigma
```

Once the weight selection is complete, it remains to design the LQG regulator and to recover its full-state loop transfer function with a properly designed Kalman-Bucy filter. This can also be done very conveniently with HONEY-X commands. The regulator design and singular value plot, for example, can be accomplished with a user macro

>lqr A2 9.0 (32)

The HONEY-X commands which make up this macro are defined in Table 3. The arguments specify that system "A2" be used for the design and display with a  $\rho$  value of 9.0. From equation (27) and Figure 4C, this value should produce crossover between 5-10 rad/sec. Figure 5A shows the resulting full state loop transfer function. Note that the low frequency gain duplicates the shape of Figure 4C and that 10 rad/sec bandwidth is consistent with our stability robustness requirements.

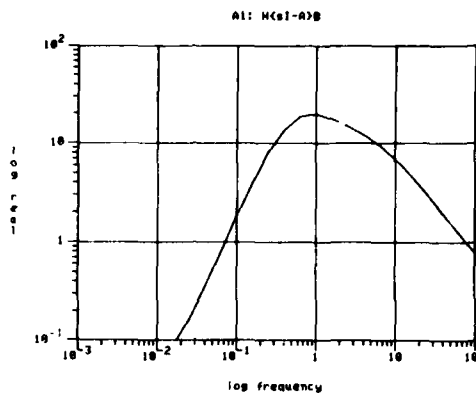
A sequence of KBF's with measurement  $\theta'$  is next used to recover the above loop transfer at point (ii). The filter design parameters are the duals of equation (8), namely

$$\begin{aligned} M &= qBB^T & q &\Rightarrow \infty \\ N &= \mu I & \mu &> 0 \text{ fixed} \end{aligned} \quad (33)$$

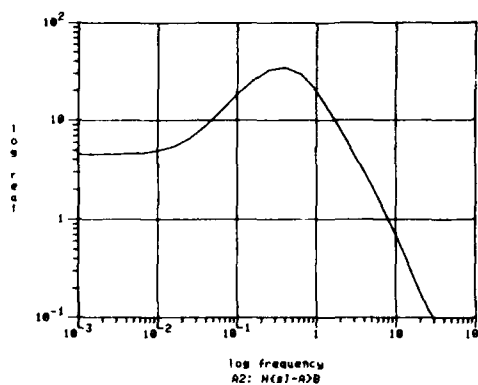
---

\*In Multics Command language, "A1" is replaced by "&l" and the entire list (28) is saved as an executive command: lqq\_weight.ec. In order to avoid later name conflicts from repeated executions, the exec com should also delete names x.a, x.b, x.c and files x.g, x.g.sigma as part of each execution. The resulting macro is shown in Table 2.]

A. Pitch  
Rate



B. Pitch  
Attitude



C. Filtered  
Pitch  
Attitude

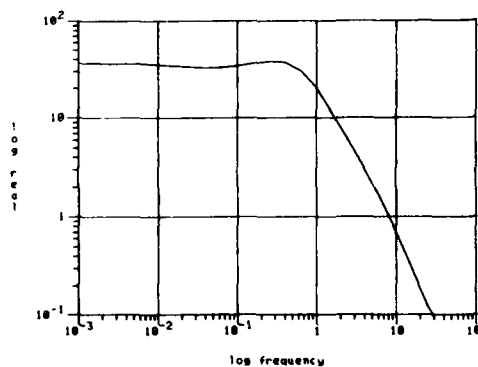


Figure 4. Weighting Selection for Pitch CAS

TABLE 3. MACRO lqr

```
&command_line off
ec matsid &l.r [cdim &l.b],&2
ec dlqgain &l
addname &l.a x.a
addname &l.b x.b
addname &l.k x.c
ec cisab x &3
ec matsvd x.g 1
ec matplt x.g.sigma/&l: Kc(sI-A)B  Rho = &2/ lwlr &4
dn x.a x.b x.c
delete x.g x.g.sigma
```

Again, a single user macro accomplishes these operations:

```
>lqr_recovery A2 1.0 1.0 (34)
```

The commands in this macro are given in Table 4. The arguments here specify that system "A2" be used to design the KBF with  $u=1.0$  and  $q=1.0$ . The macro also displays the resulting LQG loop transfer function:

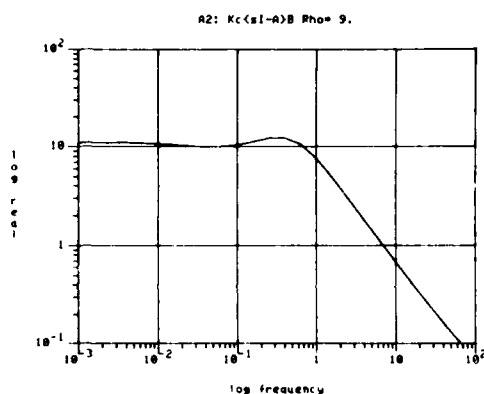
$$KG(s) = \{K_C(sI-A+BK_C+K_F C)^{-1}K_F\} C(sI-A)^{-1}B \quad (35)$$

Macro "lqr-recovery" was invoked several times with sequentially larger values of  $q$ . The results are shown in Figure 5B. Note that the LQR function,  $KG(s)$ , does indeed approach the full state design and that the value  $q=10^4$  produces adequate recovery over the required frequency range. Hence, Figure 4B represents an adequate minimal CAS design.

TABLE 4. MACRO lqr\_recovery

```
&command_line off
ec matsid &l.n [rdim &l.c], &2
ec matscl &l.b &l.gamma &3
ec dkalman &l
ec matmpy &l.kf &l.c x.t
ec matsub &l.a x.t x.a
ec matmpy &l.b &l.k x.t
ec matsub x.a x.t x.a
addname &l.kf x.b
addname &l.k x.c
ec cisab x &4
ec matmpy x.g &l.g &l.kg
ec matsvd &l.kg l
ec matplt &l.kg.sigma/&l: K(s)G(s) Mu=&2 q=&3**2/lwlr &5
dn x.b x.c
delete x.g x.a x.t
```

A. Full State Loop Transfer



B. Loop Transfer Recovery

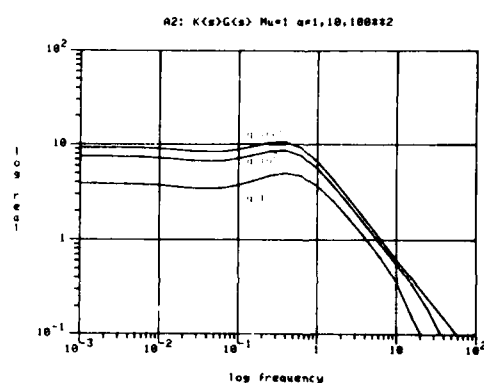


Figure 5. CAS Design Sequence

### An Angle-of-Attack Limiter

For highly maneuverable aircraft, the basic CAS designed above must usually be augmented with "boundary limiting controllers" for angle-of-attack and/or normal acceleration. Angle-of-attack limits are imposed to avoid stall, buffet, extreme drag increases and loss of control. Normal acceleration limits are based on structural considerations. Boundary limiting controllers are designed as regulators to independently control either angle-of-attack or normal acceleration. Transitions between these limiters and the normal CAS involve switching logic which is discussed in [20].

The design of boundary limiters can be achieved with the same LQG used to design the pitch CAS in the previous section. Weighting the integral of angle-of-attack, for example, and adjusting the parameter  $\rho$  results in the LQR loop gain shown in Figure 6A. This satisfies our performance objectives and is consistent with bandwidth limitations. Using integrated angle-of-attack as the sole measurement, however, the sequence of KBF's in Figure 6 fails to recover the full state loop transfer function. Subsequent evaluation of the  $\alpha/\delta$  transfer function's zeros with the HONEY-X command

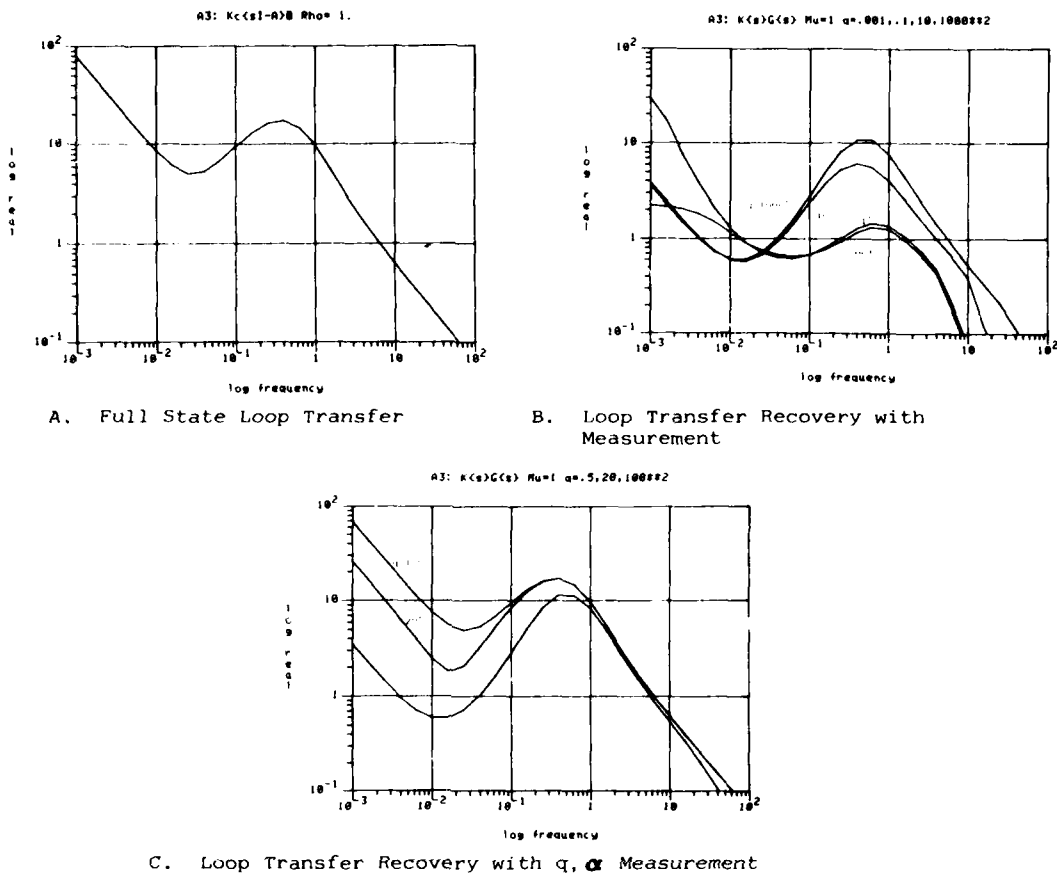
>zeros A3 (36)

revealed the presence of a non-minimum phase zero at

$s = +0.195 \text{ rad/sec}$  (37)

This zero violates the minimum-phase assumption in Section 2. The results here show that this assumption is not a technical one. Rather, non-minimum phaseness imposes fundamental design performance constraints.

The loop properties can be recovered, however, by adding other measurements to the KBF to obtain a minimum phase plant. Adding a pitch rate measurement, in this case, is sufficient. The LQR loop gain,  $T(s)$ , and the function  $KG(s)$  for several KBF iteration using the two measurements is shown in Figure 6C. Note that full state transfer function recovery has been restored.



A. Full State Loop Transfer

B. Loop Transfer Recovery with Measurement

C. Loop Transfer Recovery with  $q, \alpha$  Measurement

Figure 6. Angle-of-Attack Limiter Design Sequence

#### Decoupled Flight Path Control (DFPC)

This section discusses our final example, an advanced multi-input/multi-output control law for DFPC. Design of these flight modes is a subject of continuing research interest.

#### DFPC Objectives:

With the advent of digital flight control and the availability of multiple surfaces on new fighters there is interest in designing control modes for precision flight path control. These modes have the objective of decoupling attitude from flight path motions. They offer precise control for certain tasks as well as the possibility of new tactics for the advanced fighters. Three common decoupled flight path control objectives are [21]:

- Vertical Translation

Vertical velocity control at constant  $\theta$  ( $\alpha$  varies). Attitude remains constant as the velocity vector rotates.

- Pitch Pointing

Attitude Control at constant flight path angle (i.e.,  $\theta = \alpha$ ). Note that the velocity vector does not rotate. The angle of attack is varied at a constant normal acceleration.

- Direct Lift

Flight path control at constant angle of attack (i.e.,  $\gamma = \theta$ ). This mode produces a normal acceleration response without changing angle-of-attack.

All three of these modes are illustrated in Figure 7. They can all be achieved by a single multivariable controller which provides independent regulation of pitch attitude and angle-of-attack. For example, vertical translation can be accomplished by commanding an  $\alpha$ -profile while holding  $\theta$  constant. Similarly, pitch pointing can be accomplished by simultaneously commanding  $\alpha$  and  $\theta$ , and direct lift by commanding  $\theta$  with  $\alpha$  constant.

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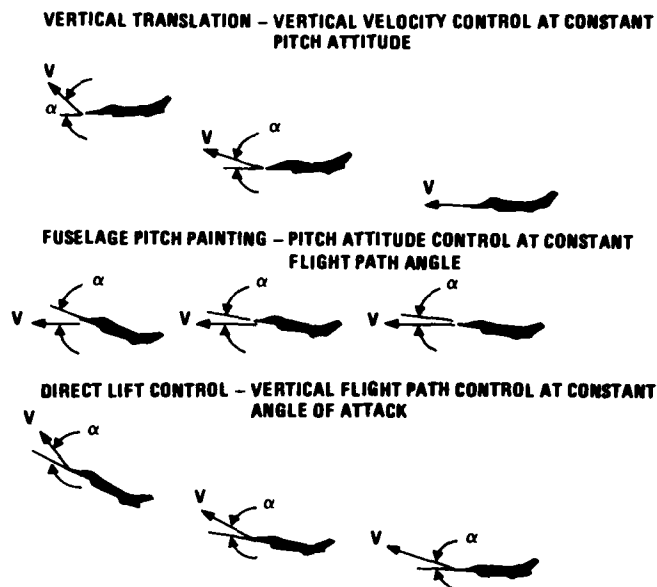


Figure 7. DFPC Modes

As discussed above, the basic objective of a decoupled flight path control loop is to provide independent control of angle-of-attack and pitch attitude with approximately equivalent speeds of response. The controls available for this function consist of the standard elevator ( $\delta_e$ ) and a forward mounted canard ( $\delta_c$ ). As before, uncertainties in the actuators, structural resonances and time delays in the RAV facility restrict the loop bandwidth to approximately 10 rad/sec.

Because loop transfer properties are desired specifically at the measured outputs, (i.e., point (i) in Figure 2) the LQG-based design procedure begins by designing a KBF with desired loop-transfer properties. Letting the first candidate parameters for this design be

$$M = \Gamma \Gamma^T \text{ with } \Gamma = B \quad (38)$$

$$N = \mu I \quad \text{with } \mu = 1$$

and appending integrators to  $\theta$  and  $\alpha$  at the outset (based on our experience above), the user macro

$$> \text{kb\_weight } A4 \quad (39)$$

gives the  $\sigma$ -plots for  $C(sI-A)^{-1}\Gamma$  shown in Figure 8A. These plots exhibit wide separation between  $\underline{\sigma}$  and  $\bar{\sigma}$ . As noted in Section 2, the KBF loop transfer function will closely approximate these  $\sigma$ -values whenever  $\sigma \gg 1$ . Hence, we must expect dramatically different gains and speeds of response in two loops of the system.

Singular value separations can be alleviated by "balancing" the noise input matrix. For example, with

$$\Gamma = B(V\Sigma^{-1}) \quad (40)$$

where  $V$  is the right singular vector matrix of  $G(j\omega_i)$  and  $\Sigma$  is the corresponding matrix of singular values, the two new singular values will become balanced (identical) at  $\omega = \omega_i$ . This follows from (23). For  $\omega_i = 10^{0.5}$ , the following HONEY-X commands calculate  $V$  and  $\Sigma$ :

$$\begin{aligned} &> \text{addname } A4.a \ x.a \\ &> \text{addname } A4.b \ x.b \\ &> \text{addname } A4.gamma \ x.c \\ &> \text{cisab } x \ 0.5, 0.5, 1. \\ &> \text{matsvd } x.g \ 1.3 \\ &> \text{matprt } x.g.sigma \ 0.5, 0.5, 1. \\ &> \text{matprt } x.g.v \ 0.5, 0.5, 1. \end{aligned} \quad (41)$$

The results are

$$V = \begin{bmatrix} .962 & -.274 \\ -.274 & -.962 \\ +j.00046 & +j.0016 \end{bmatrix} \text{ and } \Sigma = \text{diag} (1.8 \quad .0077) \quad (42)$$

While  $V$  is generally complex and not rational as a function of  $\omega$ , note that its value at  $\omega = 10^{0.5}$  is nearly real. Using only the real part in (40), therefore, gives a good balancing approximation. This is verified in Figure 8B which shows the corresponding new  $\sigma$ -plots.

The full-state KBF loop transfer function,  $T(s) = C(sI-A)^{-1}K_f$ , corresponding to the weights in Figure 8B is shown in Figure 9, together with the loop transfer  $GK(s)$  for several steps of the subsequent LQR design sequence. Following Section 2, these steps use the design parameters

$$\begin{aligned} Q &= qC^TC & q &\Rightarrow \infty \\ R &= \rho I & \rho &\text{fixed} \end{aligned} \quad (43)$$

The convergence of  $GK(s)$  toward  $T(s)$  is clearly evident. It appears that the design at  $q = 10^6$  is an adequate approximation of  $T$ . Sigma-plots of its total return difference function,  $I+GK$ , and its stability robustness function  $GK(I+GK)^{-1}$  are shown in Figure 10. These verify that design objectives (1) and (3) are indeed satisfied for the multivariable DFPC system.

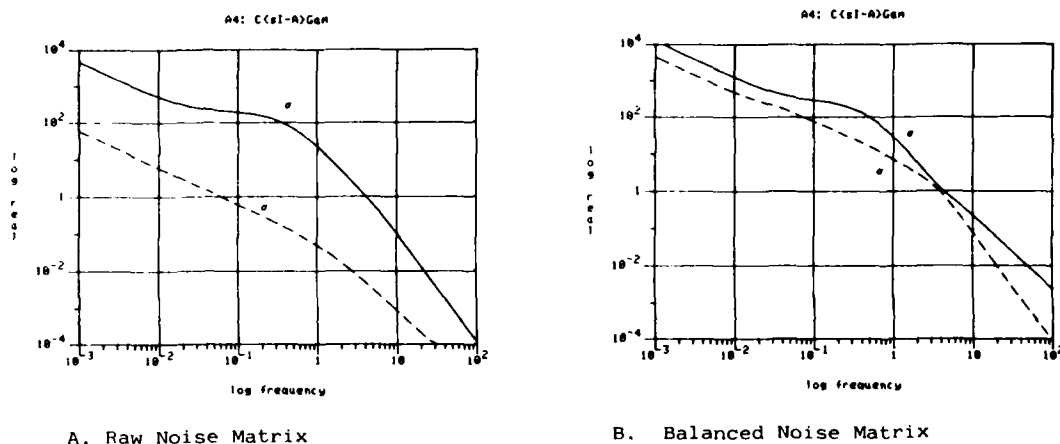


Figure 8. Weighting Selection for DFPC

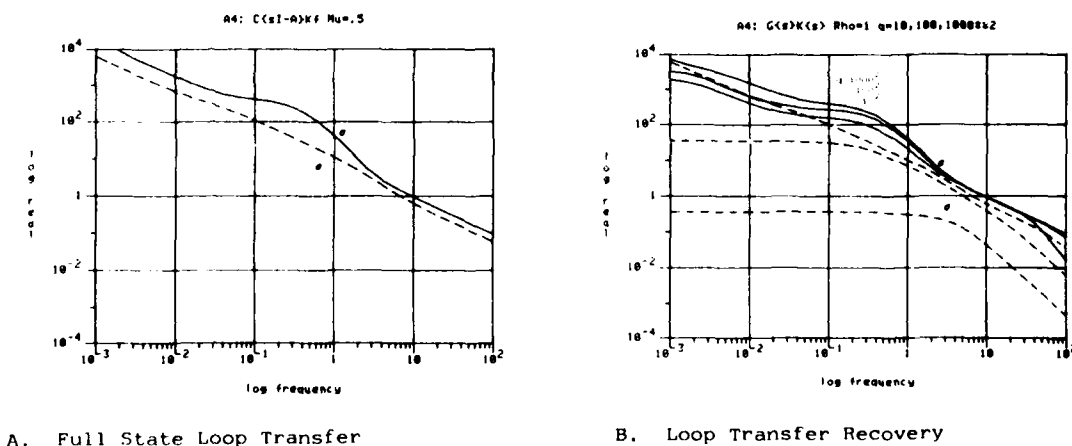


Figure 9. DFPC Design Sequence



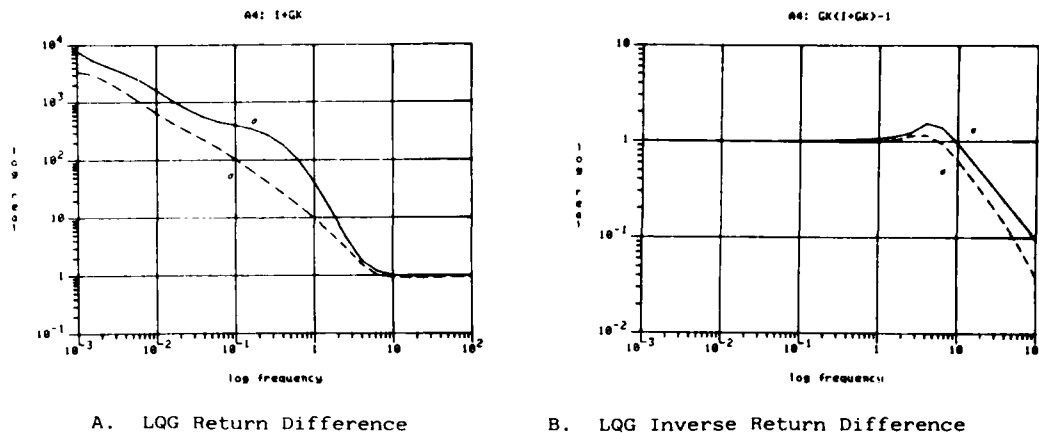


Figure 10. Close Loop DFPC Properties

## IV. CONCLUSIONS

This paper has illustrated the use of the LQG design methodology for frequency-domain-oriented multivariable feedback design. The basic frequency domain properties of LQG were reviewed, key algorithms needed for its implementation were discussed, and an interactive computer-aided design setup was introduced to carry out LQG designs effectively. These elements were illustrated with several fight control designs for highly maneuverable aircraft. While many additional design efforts need to be carried out before final judgements should be passed, the LQG procedure and its interactive computer-aided implementation appears to offer a very powerful and efficient approach to modern multivariable feedback design.

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# DESIGN OF HIGH INTEGRITY MULTIVARIABLE CONTROL SYSTEMS

by

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## SUMMARY

This paper explains the notions of (almost) controlled and conditionally invariant subspaces of linear systems and outlines their application to the synthesis of feedback compensators for disturbance rejection and to robust controller design.

## INTRODUCTION

In this lecture we will present a general introduction to the disturbance rejection and the robust controller design problems which we will view as a procedure for designing a high integrity feedback control system. The general problem set-up is the following. The model which will be used for the plant is shown in Figure 1. The plant thus accepts two types of inputs: 'unwanted' disturbances  $d$  and controls  $u$  which the designer can choose, and it produces two types of outputs: the *measured outputs*  $y$  and the *to-be-controlled outputs*  $z$ . This model suggests that the disturbances are externally generated signals. There are of course many situations in which in addition to external inputs, the disturbances also consist of parameter variations, uncertain nonlinearities, and internal noise in the system. We will lump all these perturbations which act on the nominal plant into one uncertain feedback loop which yields the situation shown in Figure 2.

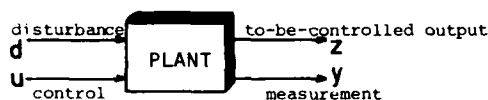


FIGURE I

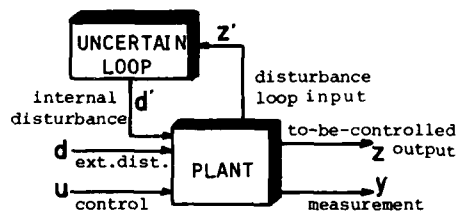


FIGURE II

In control system synthesis we are asked to design a *feedback compensator* which uses the measurement output in order to select the control input (see Figure 3) according to some design requirement which seeks to achieve desirable properties for the closed loop system. This closed loop system is shown in Figure 4(a) for the case of external disturbances and in 4(b) for the case that there is also an uncertain loop in the system.



FIGURE III

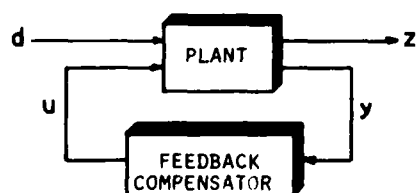


FIGURE IV(a)

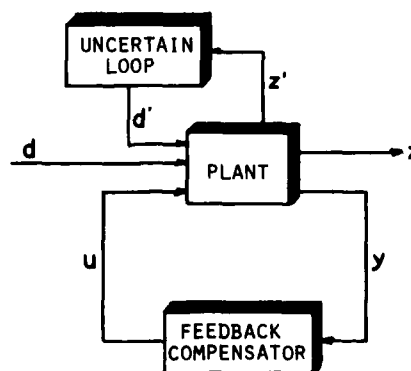


FIGURE IV(b)

In this lecture we will concentrate on what is perhaps the most natural question to ask in this context, namely the problem of *disturbance rejection*. The design philosophy is then to choose the feedback compensator in such a way that in the closed loop system (see Fig. 4(a)) the to-be-controlled output  $z$  is independent of the disturbance  $d$ . This design criterion is called *exact disturbance rejection*. We will also investigate the question of when the influence of  $d$  on  $z$  can be made arbitrarily small in some precise sense (*almost disturbance rejection*). We will return to the implications of this synthesis approach to the design of *robust controllers* i.e. feedback compensators which protect the closed loop system behavior against uncertain loops in the system (see Fig. 4(b)).

This paper contains only the outline of the theory. For proofs and more details we refer to the literature. A few words about notation:  $\mathbb{R}$  denotes the real line,  $\mathbb{C}$  the complex plane,  $\mathbb{R}^+ = [0, \infty)$ ,  $\mathbb{R}^- = (-\infty, 0]$ , and for  $n$  a positive integer  $\underline{n} = \{1, 2, \dots, n\}$ . The spectrum (i.e., the eigenvalues counting multiplicities) of the square matrix  $M$  will be denoted by  $\sigma(M)$ . We will often consider families of subspaces of a given finite dimensional vector space  $X$ . Let  $\underline{L}$  be such a family; we will say that  $\underline{L}$  is closed under addition if  $L_1, L_2 \in \underline{L} \Rightarrow L_1 + L_2 \in \underline{L}$  and closed under intersection if  $L_1, L_2 \in \underline{L} \Rightarrow L_1 \cap L_2 \in \underline{L}$ . The subspace  $\sup \underline{L}$  denotes the smallest subspace of  $X$  which contains every element of  $\underline{L}$  while  $\inf \underline{L}$  denotes the largest subspace of  $X$  contained in every element of  $\underline{L}$ . In general  $\sup \underline{L}$  and  $\inf \underline{L}$  do not belong to  $\underline{L}$ . However, the following (trivial) case yields a result which is very important to us:

**Lemma 0:** If  $\underline{L}$  is closed under addition then  $\sup \underline{L} \in \underline{L}$  and if  $\underline{L}$  is closed under intersection then  $\inf \underline{L} \in \underline{L}$ .

Let  $A: X \rightarrow X$  be a linear operator on the  $n$ -dimensional vector space  $X$  and  $L \subset X$  such that  $AL \subset L$ , then we say that  $L$  is  $A$ -invariant. We will define  $A|L$  and  $A(\text{mod } L)$  by the commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{A|L} & L \\ Q \downarrow & & \downarrow Q := \text{canonical} \\ X & \xrightarrow{A} & X \\ P \downarrow & & \downarrow P := \text{canonical} \\ X(\text{mod } L) & \xrightarrow{A(\text{mod } L)} & X(\text{mod } L) \end{array} \quad \begin{array}{l} \text{projection} \end{array}$$

If we choose a basis in  $X$  such that  $X = X_1 \oplus X_2$  with  $X_1 = L$  and  $X_2 = X(\text{mod } L)$  then in this partition  $A$  looks like  $A \approx \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  and  $A_{11} = A|L$ ,  $A_{22} = A(\text{mod } L)$ .

Let  $L_1, L_2 \in X$ . Then we will say that  $L_1$  is  $A(\text{mod } L_2)$ -invariant if  $AL_1 \subset L_1 + L_2$  and that  $L_1$  is  $A|L_2$  invariant if  $A(L_1 \cap L_2) \subset L_1$ . Furthermore  $\langle A|L_1 \rangle = L_1 + AL_1 + \dots + A^{n-1}L_1$  and  $\langle L_1|A \rangle = L_1 \cap A^{-1}L_1 \cap \dots \cap A^{n-1}L_1$  denote respectively the smallest  $A$ -invariant subspace containing  $L_1$  and the largest  $A$ -invariant subspace contained in  $L_1$ . In terms of the system  $\dot{x} = Ax + Bu$ ;  $y = Cx$ ,  $R = \langle A|\text{im } B \rangle$  is the reachable subspace, while  $N = \langle \ker C|A \rangle$  is the non-observable subspace (im and ker denote image (range) and kernel (nullspace) respectively).

Let  $f: T \rightarrow X$  with  $T = \mathbb{R}, \mathbb{R}^+$  or more generally an arbitrary interval in  $\mathbb{R}$ , and  $X$  a normed finite dimensional vector space. Then we will say that  $f \in L_p$  with  $1 \leq p < \infty$  if

$$\|f\|_{L_p} := \begin{cases} \left( \int_T \|f(t)\|^p dt \right)^{1/p} < \infty & \text{for } 1 \leq p < \infty \\ \text{ess sup}_t \|f(t)\| < \infty & \text{for } p = \infty \end{cases}$$

The notation a.c. stands for 'absolutely continuous' and a.e. stands for 'almost everywhere' and is used here exclusively in connection with Lebesgue measure.

## MATHEMATICAL MODEL

We will consider the linear time-invariant finite dimensional plant:

$$\Sigma_p: \dot{x} = Ax + Bu + Gd; \quad y = Cx, \quad z = Hx$$

with  $x \in X = \mathbb{R}^n$ ,  $u \in U = \mathbb{R}^m$ ,  $d \in D = \mathbb{R}^q$ ,  $y \in Y = \mathbb{R}^p$ , and  $z \in Z = \mathbb{R}^l$ . The feedback compensator  $\Sigma_f$  will be taken to belong to the same category, i.e., be linear time-invariant and finite dimensional:

$$\Sigma_f: \dot{w} = Kw + Ly; \quad u = Mw + Fy$$

The closed loop system has  $d$  as exogenous input and  $z$  as output and is described by:

$$\Sigma_{cl}: \dot{x}^e = A^e x^e + G^e d; \quad z = H^e x^e$$

with  $x^e = (x, w)$ , and  $(A^e, G^e, H^e)$  matrices obtainable from  $(A, B, G, C, H)$  and  $(K, L, M, F)$  in a straightforward manner:

$$A^e = \begin{bmatrix} A + BFC & BM \\ LC & K \end{bmatrix}, \quad G^e = \begin{bmatrix} G \\ 0 \end{bmatrix}, \quad H^e = \begin{bmatrix} H & 0 \end{bmatrix}$$

The solution  $z$  to  $\Sigma_{cl}$  is given by the variation of constants formula

$$z(t) = H^e e^{A^e t} x^e(0) + \int_0^t H^e e^{A^e(t-\tau)} G^e d(\tau) d\tau$$

and shows the additive influence on  $z$  of the initial condition  $x^e(0)$  and the disturbance  $d(\cdot)$ . We will denote the closed loop impulse response by  $\underline{w}$  and the closed loop transfer function by  $\underline{G}_{cl}$ . Hence

$$\underline{w}: \mathbb{R}^+ \rightarrow \mathbb{R}^{l \times q}; \quad \underline{w}(t) = H^e e^{A^e t} G^e$$

$$\underline{G}_{cl}: \mathbb{R} \rightarrow \mathbb{R}^{l \times q}; \quad \underline{G}_{cl}(s) = H^e (Is - A^e)^{-1} G^e$$

### EXACT DISTURBANCE REJECTION

In this context the exact disturbance rejection problem may be formulated as follows:

(DDPM): The disturbance decoupling problem by measurement feedback: Given  $\Sigma_p$ , does there exist  $\Sigma_f$  such that  $\underline{w} = \underline{0}$ ? If so, find algorithms for computing such a  $\Sigma_f \approx (K, L, M, F)$  from  $\Sigma_p \approx (A, B, G, C, H)$ .

Motivated in part by the above problem, there have been some important new developments in the area of linear systems since about 1970. These center around a number of new concepts. First controlled invariant and controllability subspaces (which are the key concepts in the book of Wonham [1] who did much of the original work in this area), further the conditionally invariant and complementary observability subspaces (see [2] and for a recent fully developed treatment [3]) and finally, the 'almost' versions of these concepts [4,5]. These ideas still play a very important role in the research in system theory and numerous relevant papers keep appearing.

Consider the linear system

$$\Sigma: \dot{x} = Ax + Bu$$

We will denote by  $\Sigma_x(A, B)$  all possible state trajectories generated by  $\Sigma$ . Formally,  $\Sigma_x(A, B) := \{x: \mathbb{R} \rightarrow X | x \text{ is a.c. and } \exists u: \mathbb{R} \rightarrow U \text{ such that } \dot{x}(t) = Ax(t) + Bu(t) \text{ a.e.}\}$ . Equivalently,  $\Sigma_x(A, B) = \{x: \mathbb{R} \rightarrow X | x \text{ is a.c. and } \dot{x}(t) - Ax(t) \in \text{im } B \text{ a.e.}\}$ . If there is no chance for confusion we will denote  $\Sigma_x(A, B)$  by  $\Sigma_x$ .

**Definition 1:** A subspace  $V \subset X$  is said to be a *controlled invariant subspace* if

$$\forall x_0 \in V, \exists \underline{x} \in \Sigma_x \text{ such that } x(0) = x_0 \text{ and } x(t) \in V, \forall t. \text{ (see Fig. 5)}$$

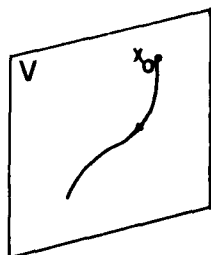


FIGURE V: CONTROLLED INVARIANCE

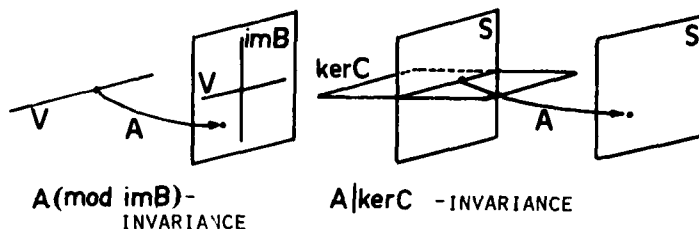


FIGURE VI

Let  $\underline{V}$  denote the set of all invariant subspaces, and  $\underline{V}(K)$  those contained in a given subspace  $K$  of  $X$ . The following property, essentially trivial, is crucial in applications.

**Proposition 1:**  $\underline{V}$  is closed under subspace addition (i.e.,  $V_1, V_2 \in \underline{V} \Rightarrow V_1 + V_2 \in \underline{V}$ ). Consequently,  $\sup \underline{V}(K) = : V_K^* \in \underline{V}(K)$ .

Equivalent properties for controlled invariance are:

- (i)  $V \in \underline{V}$  (holdability by open loop control laws)
- (ii)  $\exists F$  such that  $(A+BF)V \subset V$  (holdability by feedback controls: if we use the control law  $u=Fx$  on  $\Sigma$  then we obtain the closed loop 'flow'  $\dot{x} = (A+BF)x$  and  $x(0) \in V$  then yields  $x(t) = e^{(A+BF)t}x(0) \in V$ )
- (iii)  $AV \subset V + \text{im } B$  ( $A(\text{mod } \text{im } B)$ -invariance (see Fig. 6))

### (DDP)

Let us now consider our disturbance rejection problem with full state feedback, i.e. the problem whether there exists a feedback matrix  $F$  such that  $u = Fx$  will decouple the disturbance  $d$  from  $z$  in  $\dot{x} = Ax + Bu + Gd$ ;  $z = Hx$ . This problem is denoted by (DDP). Its solution is:

**Theorem 1:**  $\{(DDP) \text{ is solvable}\} \Leftrightarrow \{\text{im } G \subset V_{\ker H}^*\}$

The synthesis for (DDP) goes as follows:

Data: A, B, G, H

Feasibility Computation: Compute  $V_{\ker H}^*$  and verify whether  $\text{im} G \subset V_{\ker H}^*$

Feedback Gain Computation: Compute  $F$  such that  $(A + BF)V_{\ker H}^* \subset V_{\ker H}^*$   
Then  $u = Fx$  will solve (DDP).

### (DDEP)

In order to get observations into the picture, we will introduce the so-called conditionally invariant subspaces. Consider therefore the linear system

$$\Sigma': \dot{x} = Ax; y = Cx$$

Now let  $S$  be a subspace of  $X$  and consider the trajectories  $\Sigma'_x$  modulo  $S$ . In other words if  $P$  is a (surjective) map with  $S = \ker P$  we consider the output  $w = x(\text{mod} S) = Px$ .

Definition 2:  $S$  is called a *conditionally invariant subspace* if there exist  $K, L$  such that  $\dot{w} = Kw + Ly$ .

The conditionally invariant subspaces behave completely dually to controlled invariant subspaces. Let  $\underline{S}$  denote the set of all conditionally invariant subspaces, and  $\underline{S}(K)$  those containing a given subspace  $K$  of  $X$ . The following property, essentially trivial, is crucial in applications:

Proposition 2:  $\underline{S}$  is closed under subspace intersection (i.e.  $S_1, S_2 \in \underline{S} \Rightarrow S_1 \cap S_2 \in \underline{S}$ ).  
Consequently  $\inf \underline{S}(K) =: S_K^* \in \underline{S}(K)$ .

This  $S_K^*$  plays a very important role in applications and starting from  $A, C$  and  $G$ , there exist effective algorithms for computing  $S_{\text{im} G}^*$  (see Appendix).

Equivalent statements for conditional invariance are:

- (i)  $S \in \underline{S}$  ( $x(\text{mod} S)$  is tractable from  $y$ )
- (ii)  $\exists L'$  such that  $(A + L'C)S \subset S$  ( $S$  can be made invariant by output injection)
- (iii)  $A(S \cap \ker C) \subset S$  ( $S$  is  $A\ker C$ -invariant (see Fig. 6))

Actually from (ii) it is possible to compute the matrices  $K$  and  $L$  which allow us to track  $w = x(\text{mod} S) = Px$  from  $y$  via  $\dot{w} = Kw + Ly$ . This may be seen from the commutative diagram

$$\begin{array}{ccccc} & X & \xrightarrow{A + L'C} & X & \\ C \swarrow & & & & \searrow P \\ Y & & & & \text{canonical projection} \\ & \downarrow P & & \downarrow P & \\ & L'X(\text{mod} S) & \xrightarrow{K} & X(\text{mod} S) & \end{array}$$

In terms of matrices  $K$  is defined by  $KP = P(A + L'C)$  and  $L = PC$ .

An immediate application of the notion of conditionally invariant subspaces is to the *disturbance decoupled estimation problem* (DDEP). The problem (see Fig. 7) is as follows: given the plant  $\Sigma_P: \dot{x} = Ax + Gd; y = Cx, z = Hx$  with  $d$  the disturbance,  $y$  the measured output, and  $z$  the to-be-estimated output. We would like to construct an observer  $\Sigma_O: \dot{\hat{w}} = K\hat{w} + Ly; \hat{z} = M\hat{w} + Fy$ , which processes the measurements in order to produce an estimate  $\hat{z}$  of  $z$ , such that, with the initial conditions  $\hat{x}(0) = 0$ , and  $\hat{w}(0) = 0$ , the estimation error  $e = z - \hat{z}$  is independent of  $d$ , i.e. such that the transfer function  $d \rightarrow e$  is zero.

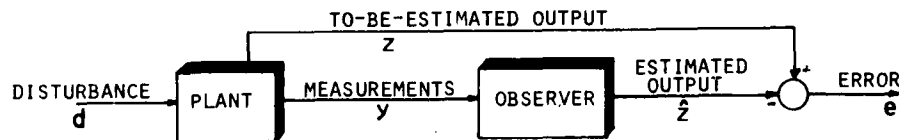


FIGURE VII

Theorem 2:  $\{(DDEP) \text{ is solvable}\} \Leftrightarrow \{S_{\text{im} G}^* \cap \ker C \subset \ker H\}$

The synthesis of the disturbance decoupled observer goes as follows:

Data: A, G, C, H

Feasibility computation: Compute  $S_{\text{im} G}^*$  and verify whether  $S_{\text{im} G}^* \cap \ker C \subset \ker H$

Observer Gain Computation: Compute  $L'$  such that  $(A + L'C)S_{\text{im} G}^* \subset S_{\text{im} G}^*$ . Let  $K$  and  $L$  be as defined in the above commutative diagram:  $KP = P(A + L'C)$  and  $L = PC$  with  $P$  the canonical projection  $X \rightarrow X(\text{mod} S_{\text{im} G}^*)$ . Let  $M$  and  $F$  be such that  $H = MP + FC$  (such  $M, F$  exist because of the condition of Theorem 2). The system  $\dot{\hat{w}} = K\hat{w} + Ly; \hat{z} = M\hat{w} + Fy$  provides the desired observer.

## (DDPM)

We now turn our attention to the disturbance decoupling problem with measurement feedback as introduced in the beginning of this section. The main ideas behind the solution of this problem are as follows. Consider the following generalization of both  $A(\text{mod } \text{im} B)$  and  $A(\text{ker } C)$  - invariant subspaces for the system

$$\Sigma: \dot{x} = Ax + Bu; y = Cx$$

**Definition 3:** A subspace  $L$  of  $X$  is called  $(A, B, C)$  - invariant (if  $L \in \mathcal{L} \cap \mathcal{S}$ )

Let  $\underline{L}$  denote the family of all such subspaces. It is easy to show that  $\{\underline{L} \in \underline{L}\} \Rightarrow \{\exists F \text{ such that } (A + BFC)L \subset L\}$ . Unfortunately  $\underline{L}$  is neither closed under subspace addition nor intersection and  $\underline{L}$  lacks hence many of the nice properties of  $\underline{V}$  or  $\underline{S}$ .

Now, if we had an  $L \in \underline{L}$  wedged in between  $\text{im} G$  and  $\text{ker} H$ :  $\text{im} G \subset L \subset \text{ker} H$  then (DDPM) would be solved. Indeed, construct  $F$  such that  $(A + BFC)L \subset L$  (linear algebra) and use the memoryless feedback compensator  $u = Fy$  on  $\Sigma_F$ . Then it is easy to see that the closed loop system  $\Sigma_{cl}: \dot{x} = (A + BFC)x + Gd; z = Hx$  will be disturbance decoupled, since  $\langle A + BFC | \text{im} G \rangle \subset \text{ker} H$  obviously implies that the transfer function  $d \rightarrow z$  is zero. However, at the present time no constructive conditions for finding such an  $L$  are known. In order to get around this problem, we introduce the notion of  $(A, B, C)$  - pairs (first introduced by Schumacher - see [3] for many other applications of this idea):

**Definition 4:** A pair  $(V, S)$  with  $V \in \underline{V}$ ,  $S \in \underline{S}$  and  $V \subset S$  will be called an  $(A, B, C)$ -pair.

$(A, B, C)$  - pairs come in very handy when instead of considering  $\Sigma: \dot{x} = Ax + Bu; y = Cx$ , we consider an extension of it:  $\Sigma^e: \dot{x}^e = Ax + Bu, \dot{w} = u_a; y = Cx, y_a = w$  and consider  $\Sigma^e$  with state space  $X^e = X \oplus W$ ,  $(u, u_a)$  as control input and  $(y, y_a)$  as observed output. Let  $P$  be the projection  $P: X^e \rightarrow X$ . Then it is easy to show the following simple

**Proposition 3:**  $\{V^e \in \underline{V}^e\} \Rightarrow \{P V^e \in \underline{V}\}$  and  $\{S^e \in \underline{S}^e\} \Rightarrow \{S^e \cap X \in \underline{S}\}$

From this proposition it follows immediately that every  $L_e \in \underline{L}_e$  generates with  $(PL_e, L_e \cap X)$  and  $(A, B, C)$  - pair. The nice thing however is that every  $(A, B, C)$  - pair is obtained this way. In other words given an  $(A, B, C)$  - pair  $(V, S)$ , there exists an extension  $\Sigma^e$  of  $\Sigma$  (the dimension of  $W$  in the extension need never be larger than  $\dim V - \dim S$ ) and an  $L_e \in \underline{L}_e$  such that  $PL_e = V \supset S = L_e \cap X$ . In order to see this, take  $W = V(\text{mod } S)$  and let  $Z \subset V \oplus W$  be such that  $\dim Z = \dim W$ ,  $Z \cap V = Z \cap W = \{0\}$ , and  $PZ \oplus S = V$ . Then  $L_e = Z \oplus S$  satisfies all the required properties. From this it may be seen that any  $(A, B, C)$  - pair  $(V, S)$  wedged in between  $\text{im} G$  and  $\text{ker} H$ :  $\text{im} G \subset S \subset V \subset \text{ker} H$ , yields a solution to (DDPM) as follows: let  $L_e$  be such that  $(V, S) = (PL_e, L_e \cap X)$ . Consider the extended plant  $\Sigma^e: \dot{x}^e = Ax + Bu + Gd, \dot{w} = u_a; y = Cx, y_a = w, z = Hx$ , written compactly as  $\dot{x}^e = A^e x^e + B^e u^e + G^e d; y^e = C^e x^e, z = H^e x^e$ . Obviously  $L^e = (A^e, B^e, C^e)$  - invariant and, since  $\text{im} G = \text{im} G^e \subset S = L^e \cap X \subset L^e \subset PL^e \oplus W = V \oplus W \subset \text{ker} H \oplus W$ , it is wedged in between  $\text{im} G^e$  and  $\text{ker} H^e$ . Hence there exists  $F^e: V^e \rightarrow U^e$  such that  $(A^e + B^e F^e C^e) L^e \subset L^e$ . This feedback control will disturbance decouple in the extended system. The beautiful thing however, is that this static feedback control law in the extended plant, is dynamic feedback in the non - extended plant  $\Sigma$ . Indeed, let  $F^e: Y_a \oplus Y \rightarrow U_a \oplus U$  be partitioned as  $F^e = \begin{bmatrix} K & L \\ M & F \end{bmatrix}$ . Then writing out the feedback law  $u^e = F^e y^e$  gives us the dynamic compensator  $\Sigma_c: \dot{w} = Kw + Ly, u = Mw + Fy$  which will decouple the disturbance  $d$  from the controlled output  $z$ . Formally stated:

**Theorem 3:**  $\{(DDPM) \text{ is solvable}\} \Leftrightarrow \{S_{\text{im} G}^* \subset V_{\text{ker} H}^*\}$

Conceptualizing this into an algorithm gives:

**Data:**  $A, B, G, C, H$

**Feasibility Computation:** Compute  $S_{\text{im} G}^*$  and  $V_{\text{ker} H}^*$ . Verify if  $S_{\text{im} G}^* \subset V_{\text{ker} H}^*$ .

**Compensator Gain Computation:** Take  $\dim W = \dim V_{\text{ker} H}^* - \dim S_{\text{im} G}^*$ . Pick  $L^e \subset X^e \oplus W$  such that  $PL^e = V_{\text{ker} H}^*$  and  $L^e \cap X = S_{\text{im} G}^*$ . Compute  $F^e$  such that  $(A^e + B^e F^e C^e) L^e \subset L^e$ . Then  $F^e = \begin{bmatrix} K & L \\ M & F \end{bmatrix}$  yields the desired disturbance rejecting compensator.

## (DDPMS) AND (DDPMPP)

In the disturbance decoupling problem considered up to now, we have concentrated on the problem of making the influence of the disturbance on the controlled output equal to zero. Thus we know when and how we can make the second term in the response

$$z(t) = H^e A^e t x^e(0) + \int_0^t H^e A^e(t-\tau) G^e d(\tau) d\tau$$

equal to zero. Of course, in applications we will also have to worry about the transient response  $H^e A^e t x^e(0)$ . It is natural to do this by imposing some internal stability requirements on the closed loop system. A general formulation for this is to introduce  $\mathcal{C} \subset \mathcal{P}$  (the 'good' part of the complex plane:  $\mathcal{C}$  is any non-empty subset of  $\mathcal{C}$  with at least one point on the real axis. If we require simple asymptotic stability, we would have  $\mathcal{C} = \{s \in \mathcal{C} | \text{Re}(s) < 0\}$ ), and require that the spectrum of  $A^e$ ,  $\sigma(A^e)$ , be in  $\mathcal{C}$ . This yields:

(DDPMS): The disturbance decoupling problem with measurement feedback and stability: Given  $\Sigma_p$  and  $\Sigma_g$ , does there exist  $\Sigma_c$  such that  $\underline{w} = \underline{0}$  and  $\sigma(A^e) \subset \mathbb{C}_g$ ? If so, find algorithms for computing  $\Sigma_c$  from  $\Sigma_p$  and  $\Sigma_g$ .

If we require (DDPMS) to be solvable for all  $\mathbb{C}_g$ , then we speak of (DDPMPP): The disturbance decoupling problem with measurement feedback and pole placement.

Unfortunately, we cannot enter into details about these problems. They are treated for instance in [6], where other relevant references may be found. We simply state the result:

**Theorem 4:**  $\{(DDPMS) \text{ is solvable}\} \Leftrightarrow \{S_{g, \text{im}G}^* \subset V_{g, \text{ker}H}^* \text{ and } \{N_{\text{im}G}^* \subset R_{\text{ker}H}^*\}$

here,  $V_{g, \text{ker}H}^*$  (resp.  $R_{\text{ker}H}^*$ ) denotes the supremal stabilizable (relative  $\mathbb{C}_g$ ) controlled invariant (resp. controllability) subspace contained in  $\text{ker}H$ , while  $S_{g, \text{im}G}^*$  (resp.  $N_{\text{im}G}^*$ ) denotes the infimal complementary detectability (relative  $\mathbb{C}_g$ ) (resp. observability) subspace containing  $\text{im}G$ . These subspaces are computable by means of finite algorithms (see Appendix) and once the conditions are verified, the computation of the compensator gains may be carried out with the aid of a pole placement routine.

### ALMOST CONTROLLED INVARIANT SUBSPACES

We consider again the system  $\dot{x} = Ax + Bu$ . As we have seen, a subspace is said to be a controlled invariant subspace if starting in it, we can stay in it by choosing the control properly. If we only require to remain arbitrarily close to the subspace, then we arrive at an almost controlled invariant subspace (see Fig. 8). Its formal definition is:

**Definition 5:** A subspace  $V_a \subset X$  is said to be an *almost (controlled) invariant subspace* if  $\forall x_0 \in V_a$  and  $\varepsilon > 0$ ,  $\exists x \in \Sigma_x$  such that  $x(0) = x_0$  and  $d(x(t), V_a) < \varepsilon$ ,  $\forall t$ , where  $d(x(t), V_a) := \inf_{v \in V_a} \|v - x(t)\|$ . A subspace  $R_a \subset X$  is said to be an *almost controllability subspace* if  $\forall x_0, x_1 \in R_a$ ,  $\exists T > 0$  such that  $\forall \varepsilon > 0$ ,  $\exists x \in \Sigma_x$  with the properties that  $x(0) = x_0$ ,  $x(T) = x_1$ , and  $d(x(t), R_a) < \varepsilon$ ,  $\forall t$ .

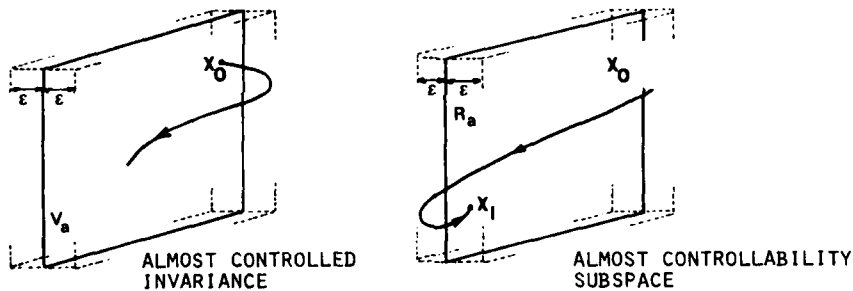


FIGURE VIII

Let  $V_a, R_a$  denote the set of all almost controlled invariant and almost controllability subspaces and  $V_a(K), R_a(K)$  those contained in a given subspace  $K$  of  $X$ . It is easy to see that  $V_a(K)$  and  $R_a(K)$  are closed under subspace addition and hence  $\sup R_a(K) =: R_{a,K}^*$ ,  $K \in R_a$  and  $\sup V_a(K) =: V_{a,K}^*$ ,  $K \in V_a$ .

In [4] we have given a large number of equivalent statements for almost invariance. The main conclusions are the following:

- (i)  $V_a \in \underline{V}_a$  (almost controlled invariance)
- (ii)  $V_a \in \underline{V} + R_a$  (every  $V_a \in \underline{V}_a$  is decomposable into a controlled invariant and an almost controllability subspace)
- (iii)  $\exists F$  and subspaces  $\text{im}B \supset B_1 \supset B_2 \supset \dots \supset B_n$  such that  $V_a = V + R_a$  where  $V \in \underline{V}$  with  $(A+BF)V \subset V$  and  $R_a \in R_a$  with  $R_a = B_1 + (A+BF)B_2 + \dots + (A+BF)^{n-1}B_n$  (this property of  $R_a$  may be interpreted as follows: if we use first feedback  $F$  and then look which states are reachable using delta functions  $\delta$  in the direction  $B_1, \delta$  in the direction  $B_2$ , etc., then the reachable set is  $B_1 + (A+BF)B_2 + \dots + (A+BF)^{n-1}B_n$ . The representation of  $R_a$  shows that with smooth approximations of  $\delta, \delta, \dots, \delta^{(n-1)}$  we can achieve such trajectories approximately)
- (iv)  $V_a \in \underline{V}_D$  := the 'distributionally controlled invariant subspaces' (see [4]:  $V_a$  is holdable using generalized functions as inputs)
- (v)  $V_a \in \underline{V}_{\text{closure}}$  (how this closure is to be understood is explained in [4]. Intuitively this means that  $\exists V_\varepsilon \in \underline{V}$  such that  $\lim_{\varepsilon \rightarrow 0} V_\varepsilon = V_a$ . Now  $\exists F_\varepsilon$  such that  $(A+BF_\varepsilon)V_\varepsilon \subset V_\varepsilon$  and, as  $\varepsilon \rightarrow 0$ ,  $F_\varepsilon \rightarrow \infty$  if  $V_a \in \underline{V}_a$  but  $\lim_{\varepsilon \rightarrow 0} F_\varepsilon V_a \notin \underline{V}$ : high gain feedback)

The theory of almost controlled invariant subspaces is very apt for studying the solvability of convolution equations. Indeed, let  $K := \ker H$  and consider the system  $\dot{x} = Ax + Bu$ ;  $z = Hx$ . If we ask when for a given  $\underline{x}(0)$  there exists  $\underline{u} : \mathbb{R}^+ \rightarrow U$  such that  $\underline{z} = H\underline{x} = \underline{0}$ , we obtain the equation

$$\underline{z}(t) = He^{At}\underline{x}(0) + \int_0^t He^{A(t-\tau)}B u(\tau) d\tau \quad (L_1)$$

or, after taking Laplace transforms

$$\underline{Z}(s) = H(Is-A)^{-1}\underline{x}(0) + H(Is-A)^{-1}B \underline{U}(s) \quad (L'_1)$$

Let us say that  $L_1$  or equivalently  $L'_1$  is approximately solvable (in the  $L_1$ -sense) if  $\forall \epsilon > 0 \exists \underline{u}$  such that  $\|\underline{z}\|_1(0, \infty) < \epsilon$ . The following holds:

**Proposition 4:**  $\{L_1 \text{ is solvable}\} \Leftrightarrow \{\underline{x}(0) \in V_{\ker H}^*\} \Leftrightarrow \{L'_1 \text{ is solvable with } \underline{U}(s) \text{ a strictly proper rational function}\}$   
 and  $\{L_1 \text{ is approximately solvable}\} \Leftrightarrow \{\underline{x}(0) \in V_{b, \ker H}^* := R_{b, \ker H}^* + V_{\ker H}^* \text{ with } R_{b, \ker H}^* := AR_{b, \ker H}^* + \text{im} B\} \Leftrightarrow \{L'_1 \text{ is solvable with } \underline{U}(s) \text{ a generalized function}\} \Leftrightarrow \{L'_1 \text{ is solvable with } \underline{U}(s) \text{ a rational function}\} \Leftrightarrow \{L'_1 \text{ is approximately solvable with } \underline{U}(s) \text{ a strictly proper rational function}\}.$

### (ADDDPM)

In this section we take a look at the problem of disturbance rejection to any arbitrary degree of accuracy. This may be formalized by introducing norms on the spaces of to-be-controlled outputs and disturbance inputs and requiring the influence of  $d$  on  $z$  to be arbitrarily small in this norm. The influence of  $d$  on  $z$  is given by:

$$\underline{z}(t) = \int_0^t \underline{W}(t-\tau) \underline{d}(\tau) d\tau$$

with  $\underline{W}$  the closed loop impulse response. Taking  $L_p$ -norms we are led to consider  $\sup_{d \neq 0} \|\underline{z}\|_{L_p(0, \infty)} / \|\underline{d}\|_{L_p(0, \infty)}$ . It is well-known that for all  $1 \leq p \leq \infty$ , this induced norm is bounded by the  $L_1$ -norm of  $\underline{W}$ , i.e. by  $\int_0^\infty \|\underline{W}(t)\| dt$ . Actually for  $p = 1, \infty$  this is the exact value, while for  $1 < p < \infty$  it is in general only an upper bound. It may be shown, however, that this fact is of no consequence on the results which follow and that for all  $p$  approximate disturbance rejection requires that we make the  $L_1$ -norm of the closed loop impulse response small. We will hence take the following problem formulation:

**(ADDDPM):** The almost disturbance decoupling problem by measurement feedback: Given  $\Sigma_p$  and any  $\epsilon > 0$ , does there exist  $\Sigma_c$  such that

$$\int_0^\infty \|\underline{W}(t)\| dt < \epsilon, \text{ where } \underline{W}: t \rightarrow H e^{A t} G_e?$$

If so, give algorithms for computing  $\Sigma_c$  from  $\Sigma_p$  and  $\epsilon$ .

It goes without saying that one can also here add internal stability requirements. However, a complete survey of what is available in this area is far beyond the scope of this lecture. We will therefore not consider further variations here.

The above problem with state measurements  $y = x$  is denoted by (ADDP) and its solvability is given in

**Theorem 5:**  $\{(\text{ADDP}) \text{ is solvable}\} \Leftrightarrow \{\text{im} G \subset V_{b, \ker H}^*\}$

The proof of this theorem also yields a conceptual algorithm for computing the required feedback gain and may be found in [4].

It is of interest to compare the striking parallel between Theorems 1 and 5. As such one expects a similar parallel to exist in the context of disturbance decoupled estimation problems. Indeed, this is the case. The duals of almost controlled invariant subspaces are the almost conditionally invariant subspaces. There are the subspaces  $S_a \subset X$  which are such that  $x(\text{mod } S_a)$  may be constructed arbitrarily closely (in a sense which is made precise in [5]). All what has been said so far about almost controlled invariance may be dualized: this yields distributionally conditionally invariant subspaces (in which  $x(\text{mod } S_a)$  may be tracked using differentiators) and there exists an infimal  $L_1$ -almost conditionally invariant subspace containing a given subspace  $K$  of  $X$ . This subspace is denoted by  $S_{b, K}^*$ . This notion is immediately applicable to the almost disturbance decoupled estimation problem (ADDEP). This problem is the same as the one in the section headed by (DDEP) but now we ask for conditions such that there exists,  $\forall \epsilon > 0$ , an observer  $\Sigma_o$  such that the  $L_1$ -norm of the impulse response  $d \mapsto e = z - \hat{z}$  is less than  $\epsilon$ . The main result is [5]:

**Theorem 6:**  $\{(\text{ADDEP}) \text{ is solvable}\} \Leftrightarrow \{S_{b, \text{im} G}^* \subset \ker H\}$

We now arrive at (ADDDPM). In this case some sort of separation theorem holds in the sense that (ADDDPM) is solvable if and only if (ADDP) and (ADDEP) are both solvable. Hence we should be able to achieve almost disturbance decoupling by state feedback and we should be able to reconstruct the to-be-controlled output arbitrarily closely from the measured output (this is the main result of [5]):

**Theorem 7:**  $\{(\text{ADDDPM}) \text{ is solvable}\} \Leftrightarrow \{\text{im} G \subset V_{b, \ker H}^* \text{ and } S_{b, \text{im} G}^* \subset \ker H\}$



## SOME SPECIAL CASES

There are a number of situations in which it is possible to conclude the solvability or the generic solvability or (DDP), (DDPM), etc. without having to go through the explicit calculations of  $V_{\ker H}^*$ ,  $V_{b, \ker H}^*$ , etc. With 'generic' solvability we mean the following: consider the dimensions of  $X, U, D, Y$  and  $Z$  as fixed but the entries of the matrices  $(A, B, C, G, H)$  as generated 'at random'. The intuitive content of the notion of generic solvability is the following: if for 'almost all'  $(A, B, G, C, H)$  the solvability (e.g. of (DDP)) holds, then we say that (DDP) is generically solvable. For the precise mathematical definition of this concept in the present framework, we refer to [4].

We have the following:

Assume:	Then generic solvability of:	holds iff:
$A, B, C$ arbitrary $H, G$ subject to $HG = 0$	(DDP) (DDPPP) (DDEP) (DDEPPP) (DDPM) (DDPMPP)	$m > l$ $m > l$ $p \geq q$ $p > q$ $m \geq l$ and $p \geq q$ $m > l$ and $p > q$
$A, B, G, C, H$ arbitrary	(ADDP) (ADDEP) (ADDPM)	$m \geq l$ $p \geq q$ $m \geq l$ and $p \geq q$

Assume:	Then solvability of:	holds iff:
	(ADDP) (ADDEP) (ADDPM)	$\text{im}G \subset \text{im}B$ $\ker C \subset \ker H$ $\text{im}G \subset \text{im}B$ and $\ker C \subset \ker H$
$R_{\ker H}^* = \{0\}$ (i.e. $H(Is-A)^{-1}B$ is left invertible)	(ADDP)	$\dim \text{im}B \geq \text{codim } \ker H$
$N_{\text{im}G}^* = X$ (i.e. $C(Is-A)^{-1}G$ is right invertible)	(ADDEP)	$\dim \text{im}G \leq \text{codim } \ker C$
$R_{\ker H}^* = \{0\}$ $N_{\text{im}G}^* = X$	(ADDPM)	$\dim \text{im}B \geq \text{codim } \ker H$ $\dim \text{im}G \leq \text{codim } \ker C$

From the above statements the importance of the conditions

$$\begin{array}{l} \# \text{ of controls} \geq \# \text{ of to-be-controlled outputs} \\ \# \text{ of measured outputs} \geq \# \text{ of disturbances} \end{array}$$

may be seen. We call this the *law of requisite variety in control action and measurement capability*. Thus good control is only possible if this variety is sufficiently large.

In 4.5 a complete analysis is made for the scalar case ( $m = q = p = l = 1$ ) where the solvability of the various disturbance decoupling problems may be decided from very simple conditions on the degrees of the polynomials involved in the transfer functions  $(u, d) \leftrightarrow (y, z)$ .

## APPLICATIONS TO ROBUST CONTROL SYNTHESIS

In this section we return to the problem of robust controller synthesis introduced in the beginning of this lecture (see Figures 2 and 4(b)). We assume that the plant is given by

$$\Sigma_P: \quad \begin{array}{l} \dot{x} = Ax + Bu + Gd + G'd' \\ y = Cx, \quad z = Hx, \quad z' = H'x \end{array}$$

with  $d$  the external disturbance,  $z$  the to-be-controlled output,  $d'$  the internal disturbance, and  $z'$  the internal disturbance loop input;  $z'$  generates  $d'$  by means of a dynamical system  $F_{\text{dist}}$  which may be nonlinear, uncertain, time-varying and which may incorporate uncertain nonlinearities, parameter variations, and other such unpleasant things.

From the (DDPM) results it follows that, if we disregard stability considerations, then it will be possible to make  $z$  insensitive to both the external and internal uncertain loop features if  $S_{\text{im}G}^* + \text{im}G' \subset V_{\ker H}^*$ .

If we require however that at the same time internal stability should hold for all finite gain  $L_p$  - input/output stable systems  $F_{dist}$ , then a sufficient condition is  $S_{imG+imG'}^* \subset V_{g,kerH}^* \cap kerH'$ . These conditions are verifiable by finite algorithms and yield a feedback compensator which may be used in order to achieve these robust control features. Note that the special cases (suitably adapted) given in the previous section are of much interest in this context as well.

The above conditions yield designs in which the same feedback compensator may be used for any  $F_{dist}$ . If we however rephrase the problem by asking when for all  $F_{dist}$  which are finite gain  $L_p$  - input/output stable with gain  $\leq K < \infty$  there should exist a feedback compensator (which may hence depend on  $K$ ) such that the closed loop systems remains stable, then we may use the theory of almost invariant subspaces and we arrive at the following conditions:

- (i) if the initial conditions are zero (hence to some extent disregarding internal stability considerations) then a sufficient condition for the existence of such a compensator are

$$S_{imG+imG'}^* \subset kerH \cap kerH'$$

and  $imG + imG' \subset V_{kerH \cap kerH'}^*$

- (ii) if we also require internal stability and take  $y = x$  the it suffices to have

$$imG + imG' \subset V_{g, kerH \cap kerH'}^* + R_{b, kerH \cap kerH'}^*$$

We close this section by referring to [7] where results on robust controller synthesis are derived in a context which is more general than the one considered here since it allows more than one uncertain loop in the system and in addition considers white noise stochastic disturbances. The results obtained in [7] use the same geometric concepts exposed in this section, but arrive at results which exploit the structure of the system in a much more subtle and effective manner.

## APPENDIX

The most basic relevant algorithms in this area are the following

$$\begin{aligned} \text{(ISA): } V_{kerH}^{k+1} &= kerH \cap A^{-1}(V_{kerH}^k + imB); & V_{kerH}^0 &= X \\ \text{(ACSA): } R_{kerH}^{k+1} &= kerH \cap (AR_{kerH}^k + imB); & R_{kerH}^0 &= \{0\} \\ \text{(ACSA)': } S_{kerH}^{k+1} &= imB + A(S_{kerH}^k \cap kerH); & S_{kerH}^0 &= \{0\} \end{aligned}$$

These algorithms attain their limits monotonically (hence in a finite number of steps  $\leq \dim X$ ) and we have:

$$\begin{aligned} \lim_{k \rightarrow \infty} V_{kerH}^k &= V_{kerH}^*; \lim_{k \rightarrow \infty} R_{kerH}^k = R_{a,kerH}^*; \lim_{k \rightarrow \infty} (V_{kerH}^k + R_{kerH}^k) = V_{a,K}^*; \\ \lim_{k \rightarrow \infty} (V_{kerH}^k \cap R_{kerH}^k) &= R_{kerH}^*; \text{ and } \lim_{k \rightarrow \infty} S_{kerH}^k = \lim_{k \rightarrow \infty} (AR_{kerH}^k + imB) = R_{b,kerH}^* \end{aligned}$$

The dual algorithms are the following:

$$\begin{aligned} \text{(CISA): } S_{imG}^{k+1} &= imG + A(S_{imG}^k \cap kerC); & S_{imG}^0 &= \{0\} \\ \text{(ACOSA): } N_{imG}^{k+1} &= imG + (A^{-1}N_{imG}^k) \cap kerC; & N_{imG}^0 &= X \\ \text{(ACOSA)': } V_{imG}^{k+1} &= kerC \cap A^{-1}(V_{imG}^k + imG); & V_{imG}^0 &= X \end{aligned}$$

These algorithms attain their limits monotonically (hence in a finite number of steps  $\leq \dim X$ ) and we have

$$\begin{aligned} \lim_{k \rightarrow \infty} S_{imG}^k &= S_{imG}^*; \lim_{k \rightarrow \infty} N_{imG}^k = N_{a,imG}^*; \lim_{k \rightarrow \infty} (S_{imG}^k \cap N_{imG}^k) = S_{a,imG}^*; \\ \lim_{k \rightarrow \infty} (S_{imG}^k + N_{imG}^k) &= N_{imG}^*; \text{ and } \lim_{k \rightarrow \infty} V_{imG}^k = \lim_{k \rightarrow \infty} (A^{-1}N_{imG}^k) \cap kerC = S_{b,imG}^* \end{aligned}$$

## REFERENCES

The following reference list contains only those mentioned in the text and does not do justice to the many contributions in this area. However, the reader will have no difficulty in obtaining via the bibliography of [1, 3, 4, or 5] other suitable entries into the literature on this topic.

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A GEOMETRIC APPROACH TO MULTIVARIABLE  
CONTROL SYSTEM SYNTHESIS

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SUMMARY

These notes present in summary fashion the essential results of the author's work: Linear Multivariable Control: A Geometric Approach (Second ed., Springer-Verlag, 1979), together with some more recent developments on nonlinear systems. Part I, Linear Systems, deals in turn with *fundamentals*, *disturbance decoupling* and *output stabilization*, *controllability subspaces*, *noninteracting control*, and *regulation and tracking*. Part II, Nonlinear Systems, summarizes structurally stable nonlinear regulation with step inputs, and concludes with a version of the Internal Model Principle for controlled sequential machines.

PART I - LINEAR SYSTEMS

1. LINEAR MULTIVARIABLE SYSTEMS. FUNDAMENTALS.

1.1 State Space Description

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0$$

$$[\text{or } x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \dots]$$

$$x = \text{state vector} \in X \approx \mathbb{R}^n$$

$$u = \text{control vector} \in U \approx \mathbb{R}^m$$

$$A: X \rightarrow X, \quad B: U \rightarrow X$$

$$\text{External control: } u = v(t)$$

$$\text{State feedback control: } u = Fx(t), \quad F: X \rightarrow U$$

$$\text{Combined control: } u = Fx + v,$$

$$\dot{x}(t) = (A+BF)x(t) + Bv(t)$$

System outputs:

$$y(t) = Cx(t), \quad C: X \rightarrow Y \approx \mathbb{R}^p \quad (\text{measured output})$$

$$z(t) = Dx(t), \quad D: X \rightarrow Z \approx \mathbb{R}^q \quad (\text{regulated output})$$

1.2 Signal Flow Graphs

$$\dot{x} = Ax + Bu$$

$$u = Fx + v$$

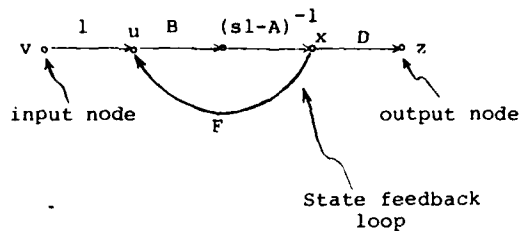
$$z = Dx$$

Take Laplace transforms:

$$\hat{x}(s) = (sI - A)^{-1} B \hat{u}(s)$$

$$\hat{u}(s) = F \hat{x}(s) + \hat{v}(s)$$

$$\hat{z}(s) = D \hat{x}(s)$$



### 1.3 Controllability, Feedback and Pole Assignment

$$\dot{x}(t) = Ax(t) + Bv(t), \quad t \geq 0$$

$$x(0) = x_0$$

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bv(\tau)d\tau$$

What states can you reach at  $t = T$  (say) by appropriate selection of  $v(\tau)$ ,  $0 \leq \tau \leq T$ ?

Let  $x_0 = 0$ .

$$\text{Have } e^{tA} = \sum_{r=1}^n \psi_r(t) A^{r-1}$$

Then

$$x(T) = \sum_{r=1}^n \int_0^T \psi_r(T-\tau) A^{r-1} B v(\tau) d\tau$$

Let  $R := \{x(T) : v(\cdot) \text{ continuous on } [0, T]\}$

Fact:  $R = \text{Im}[B \ AB \ \dots \ A^{n-1}B]$

Let  $S := \text{Im } B = BU \subset X$

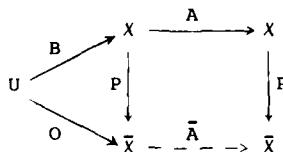
Then  $R = S + AS + \dots + A^{n-1}S$

$$=: \langle A|S \rangle \subset X.$$

$(A, B)$  is controllable if  $\langle A|S \rangle = X$ .

Suppose  $\langle A|S \rangle \neq X$ .

Let  $P: X \rightarrow \bar{X} := X / \langle A|S \rangle$ .

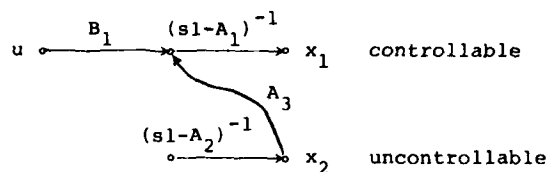


Write  $X = X_1 \oplus X_2$

$$X_1 = \langle A|S \rangle, \quad X_2 \cong \bar{X}.$$

$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

The signal flow exhibits the system decomposition.



1. Fact:  $(\forall F) \langle A+BF | \mathcal{B} \rangle = \langle A | \mathcal{B} \rangle$

i.e. Controllable subspace  $\langle A | \mathcal{B} \rangle$  is invariant under application of state feedback.

2. Let  $\dim(X) = n$ .

Consider  $\sigma(A+BF)$

$\sigma$  = symmetric set  $\Lambda_F$  of  $n$  complex numbers

Fact:  $\sigma(A+BF)$  = poles of  $(sI-A-BF)^{-1}$

Theorem ("Pole assignment"):

$(A, B)$  is controllable [i.e.  $\langle A | \mathcal{B} \rangle = X$ ]

iff

$(\forall \Lambda \subset \mathbb{C}, \text{ symmetric, } |\Lambda| = n) (\exists F: X \rightarrow U): \sigma(A+BF) = \Lambda$ .

Stabilizability

Split complex numbers  $\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b$  (g, b mean 'good', 'bad')

e.g.  $\mathbb{C}_g = \{s: \operatorname{Re} s < 0\}$

$A: X \rightarrow X$  is stable if  $\sigma(A) \subset \mathbb{C}_g$ .

$(A, B)$  is stabilizable if  $\exists F: A+BF$  is stable.

Let characteristic polynomial of  $A$  be  $\pi(s)$ .

Factor  $\pi(s) = \pi_g(s)\pi_b(s)$ , zeros of  $\pi_g$  in  $\mathbb{C}_g$ , etc.

Unstable modes of  $A$  is subspace

$X_b(A) = \operatorname{Ker} \pi_b(A)$ .

$\{X = X_b(A) \oplus X_g(A)\}$

Proposition.  $(A, B)$  is stabilizable iff

$$X_b(A) \subset \langle A | \mathcal{B} \rangle,$$

i.e. unstable modes are controllable.

## 2. DISTURBANCE DECOUPLING AND OUTPUT STABILIZATION

### 2.1 $(A, B)$ -Invariant Subspaces

$$\dot{x} = Ax + Bu + Sq$$

$$z = Dx$$

$q = q(\cdot)$  is an exogenous disturbance

Temporarily set  $u = 0$ . Let  $S = \operatorname{Im} S$ . Disturbances act in subspace

$\langle A | S \rangle$ . Output  $z(\cdot)$  is decoupled from  $q(\cdot)$  iff

$$D \langle A | S \rangle = 0.$$

Now let  $u = Fx$ .

$$\text{DDP: } \exists F: D(A+BF|S) = 0.$$

$$\text{i.e. } \exists F: D(A+BF)^{j-1}S = 0, \quad j \in \underline{n}.$$

Nasty nonlinear problem in F!

Recast in geometric form

$$\exists F: \langle A+BF|S \rangle \subset \text{Ker } D$$

The foregoing problem motivates the introduction of (A,B)-invariant subspaces.

$$\text{Let } V (= V_F) = \langle A+BF|S \rangle.$$

$$\text{Note: } (A+BF)V \subset V.$$

Problem: Given A,B, and any subspace  $V \subset X$

$$\exists F: (A+BF)V \subset V$$

Call such  $V$ : (A,B)-invariant.

Fact:  $V$  is (A,B)-invariant iff

$$AV \subset V + B.$$

Notation:  $\underline{I}(X) = \{V: AV \subset V+B\}$

$$\underline{I}(\text{Ker } D) = \{V: V \subset \text{Ker } D \text{ \& } AV \subset V+B\}$$

Note:  $0 \in \underline{I}(\text{Ker } D) \neq \emptyset.$

## 2.2 Solution of Disturbance Decoupling Problem

Proposition. DDP is solvable iff

$$\exists V: V \in \underline{I}(\text{Ker } D) \text{ \& } V \supset S.$$

{Proof. (If)

$$\text{Let } F: (A+BF)V \subset V.$$

$$\text{Then } \langle A+BF|S \rangle \subset \langle A+BF|V \rangle$$

$$= V + (A+BF)V + \dots$$

$$= V \subset \text{Ker } D. \}$$

Proposition.  $V_1, V_2 \in \underline{I}(\text{Ker } D) \implies V_1 + V_2 \in \underline{I}(\text{Ker } D)$

Proposition.  $\exists! V^* \in \underline{I}(\text{Ker } D):$

$$(\forall V) V \in \underline{I}(\text{Ker } D) \implies V \subset V^*$$

Write:  $V^* = \sup \underline{I}(\text{Ker } D).$

Theorem

DDP is solvable iff  $V^* \supset S.$

Algorithm

$$V^0 := \text{Ker } D$$

$$V^{j+1} := \text{Ker } D \cap A^{-1}(V^j + B), \quad j = 0, 1, 2, \dots$$

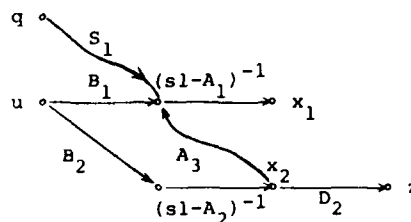
$$V^n = V^*.$$

Signal Flow

$$\text{Let } X = X_1 \oplus X_2$$

$$X_1 = V^*, \quad X_2 \cong X/V^*$$

$$A+BF = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & D_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 \\ 0 \end{bmatrix}.$$



### 2.3 Output Stabilization Problem

$$\dot{x} = Ax + Bu, \quad z = Dx$$

OSP Find  $u = Fx$  such that, for any  $x_0$ ,  $z(t) := De^{t(A+BF)}x_0$  has all its exponents in  $\mathbb{C}_g$

#### Equivalent problem

Find  $F: X \rightarrow U$  such that

$$X_b(A+BF) \subset \text{Ker } D$$

#### Theorem

Let  $V^* := \sup \underline{I}(\text{Ker } D)$ .

OSP is solvable iff

$$X_b(A) \subset \langle A|B \rangle + V^*.$$

## 3. CONTROLLABILITY SUBSPACES

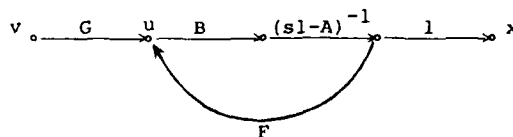
### 3.1 Controllability Subspaces: Definition and Characterizations

$$\dot{x} = Ax + Bu$$

$R \subset X$  is a controllability subspace (c.s.)

if  $\exists F: X \rightarrow U$  &  $G: U \rightarrow U$  such that

$$R = \langle A+BF | \text{Im}(BG) \rangle$$



Notation  $\underline{C}(X), \quad \underline{C}(\text{Ker } D)$

Fact:  $R_1, R_2 \in \underline{C}(\text{Ker } D) \implies R_1 + R_2 \in \underline{C}(\text{Ker } D)$

Fact:  $\exists! R^* = \sup \underline{C}(\text{Ker } D)$

Fact:  $0 \subset R^* \subset V^* \subset \text{Ker } D \subset X$ .

#### Notation

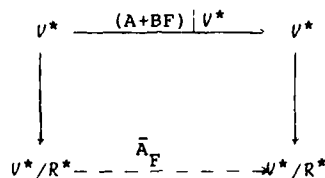
$$F(V^*) := \{F: (A+BF)V^* \subset V^*\}$$

Say: "F is a friend of  $V^*$ "

$$F(R^*) := \{F: (A+BF)R^* \subset R^*\}$$



Fact:  $\underline{F}(V^*) \subset \underline{F}(R^*)$ . "Any friend of  $V^*$  is a friend of  $R^*$ ."



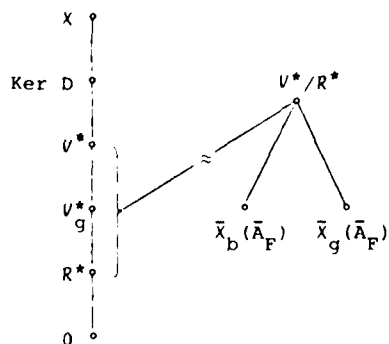
### 3.2 Transmission Zeros

Fact:  $\bar{A}_F$  is the same for all  $F \in \underline{F}(V^*)$ .

Jargon:  $\sigma(\bar{A}_F)$  is the set of transmission zeros of  $(D, A, B)$ .

### 3.3 Disturbance Decoupling Problem with Stability (DDPS)

DDPS:  $\exists F : \begin{cases} A+BF \text{ is stable, \& } \\ A+BF|S \subset \text{Ker } D \end{cases}$



$$F \in \underline{F}(V^*)$$

$$P: V^* \rightarrow V^*/R^*$$

$$V_g^* := P^{-1}(\bar{X}_g(\bar{A}_F))$$

Theorem DDPS is solvable iff

- (i)  $(A, B)$  is stabilizable
- & (ii)  $V_g^* \supset S$ .

### 4. NONINTERACTING CONTROL (DECOUPLING)

$$\dot{x} = Ax + Bu$$

$$z_i = D_i x \quad i \in \underline{k}$$

$$\text{Let } u = Fx + \sum_{i=1}^k G_i v_i$$

#### 4.1 Restricted Decoupling Problem (RDP)

$\exists F, G_i$  ( $i \in \underline{k}$ ) such that  $v_i$  controls  $z_i$  without affecting  $z_j, j \neq i$

$$\text{Let } R_i = \langle A+BF | \text{Im}(BG_i) \rangle$$

RDP can be stated:

$\exists$  c.s.  $R_i, i \in \underline{k}$ :

$$\bigcap_{i=1}^k F(R_i) \neq \emptyset, \quad D_i R_i = \text{Im } D_i, \quad R_i \subset \bigcap_{j \neq i} \text{Ker } D_j.$$

Unsolved. Add more state variables.

#### 4.2 Extended Decoupling Problem (EDP)

Adjoin auxiliary dynamics

$$\dot{x}_a = B_a u_a, \quad x_a \in X_a, \quad u_a \in U_a.$$

Extended spaces  $X_e = X \oplus X_a, \quad U_e = U \oplus U_a.$

Extended maps  $A_e = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad B_e = \begin{pmatrix} B & 0 \\ 0 & B_a \end{pmatrix}, \quad D_{ie} = (D_i \quad 0).$

EDP Find  $n_a$  and extended c.s.  $S_i$  ( $i \in \underline{k}$ ) such that

$$S_i \subset \bigcap_{j \neq i} \text{Ker } D_{je}, \quad \bigcap_{i \in \underline{k}} F_e(S_i) \neq \emptyset,$$

$$S_i + \text{Ker } D_{ie} = X_e, \quad i \in \underline{k}.$$

Theorem EDP is solvable iff, for the original RDP,

$$R_i^* + \text{Ker } D_i = X, \quad i \in \underline{k}.$$

Then can choose

$$n_a = \sum_{i=1}^k d(R_i^*) - d\left(\sum_{i=1}^k R_i^*\right).$$

#### 4.3 Generic Solvability of EDP

Regard

$$p := (D_1, \dots, D_k, A, B)$$

as a data point in  $\mathbb{R}^N$ ,

$$N = (q_1 + \dots + q_k)n + n^2 + nm.$$

Fix  $q_1, \dots, q_k, n, m$ . EDP is generically solvable if solvable for all  $p$  off some proper algebraic variety in  $\mathbb{R}^N$ .

Theorem

EDP is generically solvable iff

$$\sum_{i=1}^k q_i \leq n \quad (\text{generic noninteraction})$$

$$m \geq 1 + \sum_{i=1}^k q_i - \min_{1 \leq i \leq k} q_i \quad (\text{generic output controllability})$$

{If either condition fails, EDP is generically unsolvable!}

Then can take, generically,

$$n_a = (k-1)\left(n - \sum_{i=1}^k q_i\right)$$

#### 5. REGULATION AND TRACKING

$$\dot{x} = Ax + Bu$$

$$y = Cx \text{ (measured)}$$

$$z = Dx \text{ (to be regulated)}$$

### 5.1 Regulator Problem with Internal Stability (RPIS)

Find (dynamic) feedback from  $y$ , i.e.

$$\hat{u}(s) = \hat{F}(s)\hat{y}(s),$$

such that

- (i) All controllable, observable modes are stabilized.
- (ii) Output  $z$  is regulated, i.e. for all i.c.,  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

(Typical format:

$$\dot{x}_1 = A_1 x_1 + A_3 x_2 + B_1 u, \quad \dot{x}_2 = A_2 x_2$$

$(A_1, B_1)$  - controllable plant

$A_2$  - exosystem,  $A_3$  - disturbance coupling)

#### Admissible controls

Could try static output feedback, i.e.

$$u = Fx, \quad F = \tilde{F}C \quad \text{or} \quad \text{Ker } F \supset \text{Ker } C.$$

Intractable!

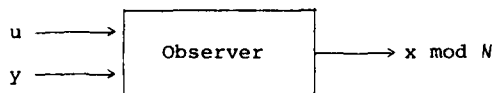
$$\text{Better: } y_{\text{new}} = C_{\text{new}} x,$$

$$\text{Ker } C_{\text{new}} = N := \bigcap_{i \geq 1} \text{Ker } C A^{i-1},$$

$$u = Fx, \quad \text{Ker } F \supset N.$$

Tractable! since  $AN \subset N$ .

This is algebraically equivalent to using a dynamic observer:



#### Internal stability

Any observable, unstable modes should be uncontrollable.

$$\mathcal{C} = \{s: \text{Re } s \geq 0\} \cup \{s: \text{Re } s < 0\}$$

$$= \mathcal{C}^+ \cup \mathcal{C}^-$$

Require:

$$\frac{\chi^+(A+BF)+N}{N} \cap \frac{\langle A|B \rangle + N}{N} = 0$$

or

$$\chi^+(A+BF) \cap (\langle A|B \rangle + N) \subset N.$$

#### Output regulation

Bury unstable modes in  $\text{Ker } D$ , namely arrange that

$$\chi^+(A+BF) \subset \text{Ker } D$$

#### RPIS

Given  $A: X \rightarrow X$ ,  $B: U \rightarrow X$ ,  $D: X \rightarrow Z$ , and  $N \subset X$  with  $AN \subset N$ .

Find  $F: X \rightarrow U$  such that

$$(i) \quad \text{Ker } F \supset N$$

$$(ii) \quad \chi^+(A+BF) \cap (\langle A|B \rangle + N) \subset N$$

$$(iii) \quad \chi^+(A+BF) \subset \text{Ker } D$$

Theorem

RPIS is solvable iff  $\exists V \subset X$  such that

- (1)  $V \subset \text{Ker } D \cap A^{-1}(V+B)$
- (2)  $X^+(A) \cap N + A(V \cap N) \subset V$
- (3)  $V \cap (\langle A|B \rangle + N) \subset N$
- (4)  $X^+(A) \subset \langle A|B \rangle + V$

5.2 Separation Theorem for RPIS

Extra dynamic compensation (beyond the observer) does not help!

5.3 Solution of RPIS: Geometric Structure

Assume first that  $N = 0$ .

$$V^* := \sup \underline{I}(\text{Ker } D), \quad R^* := \sup \underline{C}(\text{Ker } D)$$

Recall:  $\underline{F}(V^*) \subset \underline{F}(R^*)$ .

If  $A_F := A + BF$ ,  $F \in \underline{F}(V^*)$ , then  $\bar{A}_F$ , induced on  $V^*/R^*$ , is the same for all  $F \in \underline{F}(V^*)$ .

Let  $AT \subset T$  and  $AR \subset R \subset T$ .

Def.  $R$  decomposes  $T$  wrt  $A$  if  $\exists S \subset T$ ,  $R \oplus S = T$ ,  $AS \subset S$ .

Fact  $R$  decomposes  $T$  wrt  $A$

iff  $\{e.d. A|R\} \cup \{e.d. \text{ of } \bar{A} \text{ on } T/R\} = \{e.d. \text{ of } A|T\}$

iff  $A_1 V - VA_2 + A_3 = 0$  is solvable for  $V$ .

Theorem

Let  $N = 0$ . RPIS is solvable iff

- (i)  $X^+(A) \subset \langle A|B \rangle + V^*$
- (ii) In  $X/R^*$ , with  $F \in \underline{F}(V^*)$ ,

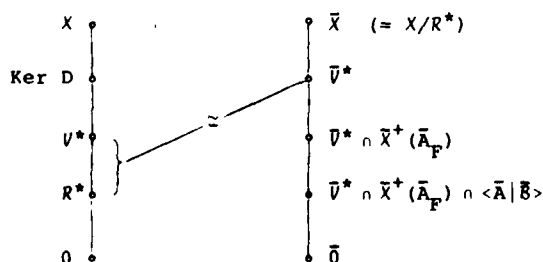
$$\frac{V^* \cap X^+(A_F) \cap \langle A|B \rangle + R^*}{R^*}$$

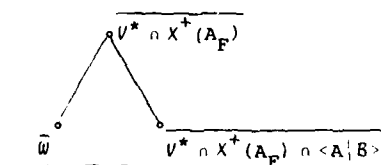
decomposes

$$\frac{V^* \cap X^+(A_F) + R^*}{R^*}$$

wrt map  $\bar{A}_F$ , induced by  $A_F$  in  $V^*/R^*$ .

(Condition needs to be checked only at one, arbitrarily selected,  $F \in \underline{F}(V^*)$ .)

Proof of Theorem (if)



by decomposability



V has required properties

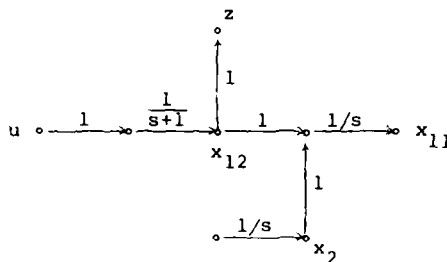
Corollary 1Assume:  $N = 0$ &  $\sigma\{\text{induced map } \bar{A} \text{ on } X/\langle A|B \rangle\} \in \mathbb{C}^+$ 

RPIS solvable iff

(i)  $\langle A|B \rangle + V^* = X$ & (ii)  $V^* \cap \langle A|B \rangle / R^*$  decomposes  $V^*/R^*$  wrt  $\bar{A}_F$  induced on  $V^*/R^*$ .Let  $X_1 := \langle A|B \rangle$ ,  $A_1 := A/\langle A|B \rangle$ , etc. $\sigma_1^* := \sigma\{\text{induced map on } V_1^*/R_1^*\}$ Corollary 2

Assumptions as in Corollary 1.

RPIS solvable provided

(i)  $\langle A|B \rangle + V^* = X$ & (ii)  $\sigma_1^* \cap \sigma(\bar{A}) = \emptyset$ .Example (RPIS not solvable)

$$R^* = 0, \quad V^* = \text{Ker } D$$

$$A_F|V^* \simeq \begin{bmatrix} 0 & 1 \\ \dots & \dots \\ 0 & 0 \end{bmatrix}, \quad V^* \cap \langle A|B \rangle \simeq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Decomposability fails!

General solution of RPIS ( $N \neq 0$ )Theorem. RPIS is solvable iff(i)  $X^+(A) \cap N \subset \text{Ker } D$ (ii) In  $\bar{X} := X/X^+(A) \cap N$ , the reduced RPIS is solvable,  
i.e.  $\exists \bar{F}: \bar{X} \rightarrow U$  such that

$$\bar{X}^+(\bar{A} + \bar{B}\bar{F}) \subset \text{Ker } \bar{D}$$

&  $\bar{X}^+(\bar{A} + \bar{B}\bar{F}) \cap \langle \bar{A}|\bar{B} \rangle = \bar{0}$ .

#### 5.4 Solution of RPIS: Linear Maps Criterion

Assume (i)  $\sigma(\text{exosystem}) \subset \mathbb{C}^+$

(ii)  $X^+(A) \cap N = 0$

(iii)  $D \langle A|B \rangle = Z$

##### Theorem

RPIS solvable iff  $\exists V \subset X$  such that

$$V \subset \text{Ker } D \cap A^{-1}(V+B),$$

$$\& \langle A|B \rangle \oplus V = X.$$

Let  $P: X \rightarrow X/\langle A|B \rangle =: \bar{X}$ .

##### Corollary

RPIS solvable iff  $\exists$  maps  $\bar{V}: \bar{X} \rightarrow X$  &  $K: \bar{X} \rightarrow U$  such that

$$AV - V\bar{A} + BK = 0, \quad DV = 0, \quad PV = 1.$$

#### 5.5 Well-Posedness and Generic Solvability

##### Coordinatization

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

$$C = (C_1 \quad C_2), \quad D = (D_1 \quad D_2).$$

$$A_1: n_1 \times n_1, \quad A_2: n_2 \times n_2, \quad B_1: n_1 \times m, \quad D_1: q \times n_1$$

$(A_1, B_1)$  controllable plant

$\sigma(A_2) \subset \mathbb{C}^+$  exosystem

$\text{Rank } D_1 = q$  (full rank)

Data point  $p = (A_1, A_3, B_1) \in \mathbb{R}^N$ ,

$$N = n_1^2 + n_1 n_2 + n_1 m.$$

Def. RPIS is well-posed at  $p$  if solvable throughout a nbhd of  $p$  in  $\mathbb{R}^N$ .

Def. Fix  $n_1, m, q$ . RPIS is generically solvable if solvable for all  $p \in \mathbb{R}^N$ , off some proper algebraic variety.

##### Theorem

RPIS is well-posed at  $p = (A_1, A_3, B_1)$  iff

$$(A_1 \otimes I_2^1 - I_1 \otimes A_2^1) \text{Ker}(D_1 \otimes I_2^1) + \text{Im}(B_1 \otimes I_2^1) = X_1 \otimes X_2^1.$$

##### Corollary

RPIS is generically solvable iff

$$m (= \text{no. of controls}) \geq q (= \text{no. of regulated outputs})$$

If  $m < q$ , no data point is well-posed.

##### Theorem

Assume also:  $(D_1, A_1)$  observable

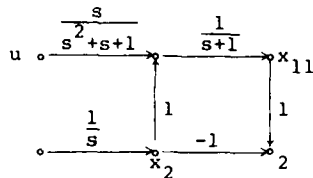
$$\& \sigma(A_1) \cap \sigma(A_2) = \emptyset.$$

RPIS is well-posed at  $(A_1, A_3, B_1)$  iff

$$H_1(\lambda) := D_1(\lambda I_1 - A_1)^{-1} B_1$$

is right-invertible (over  $\mathbb{C}$ ) at every  $\lambda \in \sigma(A_2)$

Example (RPIS solvable but ill-posed)



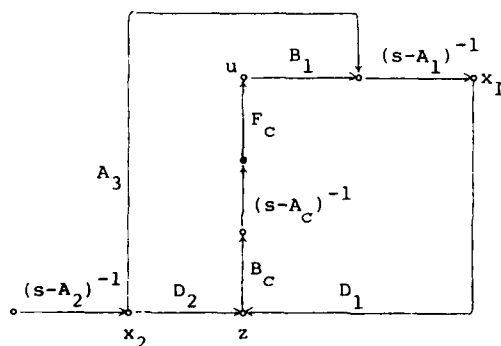
If you perturb the branch

$$\uparrow 1 \quad \text{to} \quad \uparrow 1+\epsilon$$

then RPIS becomes unsolvable.

### 5.6 Strong ('Robust') Synthesis

Assume RPIS well-posed, and  $C = D$ .



Def.  $(F_c, A_c, B_c)$  is a synthesis if

- (i)  $(F_c, A_c)$  observable and  $(A_c, B_c)$  controllable
- (ii) Loop is stable
- (iii) Output regulation holds.

Def. Let  $p = (A_1, A_3, B_1)$ . Synthesis is strong at p if properties (i) - (iii) hold throughout a nbhd of  $p$  in  $\mathbb{R}^N$ .

Theorem Let  $S_c = (F_c, A_c, B_c)$  be a synthesis.

Let  $\alpha_2(\lambda)$  be m.p. of  $A_2$ .

Let  $T: \mathbb{T} \rightarrow \mathbb{T}$  be cyclic, with m.p.  $\alpha_2$ .

Then  $S_c$  is strong at  $p$  iff

$\exists J: \mathbb{T} \otimes \mathbb{Z}' \rightarrow \mathbb{X}_c$ , injective, such that diagram commutes.

$$\begin{array}{ccc} \mathbb{T} \otimes \mathbb{Z}' & \xrightarrow{T \otimes 1_2'} & \mathbb{T} \otimes \mathbb{Z}' \\ J \downarrow & & \downarrow J \\ \mathbb{X}_c & \xrightarrow{A_c} & \mathbb{X}_c \end{array}$$

Say: " $A_c$  incorporates an internal model of  $A_2$ ".

Theorem

RPIS admits a strong synthesis at  $p$  iff it is well-posed at  $p$ .

5.7 Internal Model Principle

Assume  $y = Cx$ ,  $z = Dx$ ,

but no a priori relation between  $C$  and  $D$ .

"Synthesis" as before, but now with  $y$  as input.

"Strong" as before, but with certain variations in  $B_c$ .

Theorem

Synthesis is strong only if

(i)  $\text{Ker } C \subset \text{Ker } D$  (i.e.  $z = Ey$ , some  $E$ )

& (ii) " $A_c$  incorporates an internal model of  $A_2$  that is controlled by  $z$ ".

PART II - NONLINEAR SYSTEMS6. STRUCTURALLY STABLE NONLINEAR REGULATION WITH STEP INPUTS

Let  $X$  - smooth ( $C^\infty$ ) manifold

$TX$  - tangent bundle of  $X$

$\xi: X \rightarrow TX$  vector field, complete

System is a pair  $(X, \xi)$ .

Call  $(X, \xi)$  the regulator (system).

Bring in auxiliary system

$(V, \eta)$  - exosystem

Assume  $\exists$  surjective submersion  $\pi: X \rightarrow V$  such that we have

$$\begin{array}{ccc} X & \xrightarrow{\xi} & TX \\ \pi \downarrow & & \downarrow \pi_* \\ V & \xrightarrow{\eta} & TV \end{array}$$

Assume:  $(X, \xi)$  admits a subsystem 'induced' by  $(V, \eta)$ , in the sense:

$$\begin{array}{ccccc} & V & \xrightarrow{\eta} & TV & \\ & \downarrow v & & \downarrow v_* & \\ \text{id} \swarrow & X & \xrightarrow{\xi} & TX & \searrow \text{id} \\ & \downarrow \pi & & \downarrow \pi_* & \\ & V & \xrightarrow{\eta} & TV & \end{array} \quad (1)$$

Assume:  $v$  is a regular embedding, defined uniquely by (1), viz.

$$\xi \circ v = v_* \circ \eta, \quad \pi \circ v = \text{id}.$$

(In applications one might often require that  $v(V)$  be a stable attractor for the flow determined by  $\xi$  on  $X$ , viz.

$$(\forall x_0) x(t, x_0) \rightarrow v(V) \text{ as } t \rightarrow \infty).$$

Bring in a 'good' submanifold  $(K, \kappa)$ ,

$$K \xrightarrow{\kappa} X.$$

$K$  is where you would like to be.



(E.g.  $\kappa(K) = \delta^{-1}(z_0)$ ,

where  $\delta: X \rightarrow Z$  is an output map,  $z_0 \in Z$  a setpoint.)

### Output regulation condition

$$\nu(V) \subset \kappa(K)$$

So  $\exists$  embedding  $i: V \rightarrow K$  such that we have

$$\begin{array}{ccccc} & V & \xrightarrow{\eta} & TV & \\ \swarrow i & \downarrow \nu & & \downarrow \nu_* & \searrow i_* \\ K & & X & \xrightarrow{f_*} & TX \\ \nwarrow \kappa & & \nwarrow \kappa_* & & \end{array} \quad (2)$$

### Regulator structure

Assume

$$TX = E_0 \oplus E_c \oplus E_e$$

where the vector bundle

$E_0$  is identified with 'plant'

$E_c$  is identified with 'controller'

$E_e$  is identified with 'exosystem'

Thus

$$E_0 \oplus E_c = \text{Ker } \pi_*$$

$$\pi_* E_e = TV$$

$$f_* = f_{0*} \oplus f_{c*} \oplus f_{e*}$$

Regard  $f_{0*}, f_{c*}, f_{e*}, \pi, \kappa$  as fixed.

Consider  $f_{0*}$  variable to  $\tilde{f}_{0*}$  near  $f_{0*}$  in  $\Gamma(E_0)$ .

Assume (1) holds near  $f_{0*}$ , i.e. to each  $\tilde{f}_{0*}$  there corresponds  $\tilde{\nu}$  such that (1) holds with  $f_* = (\tilde{f}_{0*}, f_{c*}, f_{e*})$ .

Say: Regulation is structurally stable if (2) is preserved as well.

Namely

$$\begin{aligned} & (\exists \text{ nbhd } N_0 \text{ of } f_{0*}) (\forall \tilde{f}_{0*}) \\ & \tilde{f}_{0*} \in N_0 \implies (\exists \tilde{\nu}) \tilde{\nu} = \kappa \circ i. \end{aligned} \quad (3)$$

(Assume Whitney  $C^\infty$ -topology for  $\Gamma(E_0)$ )

Problem: Assuming structural stability, what can we infer about the controller  $f_{c*}$ ? Consider only the special case of

### Step Inputs

Assume  $\eta(v) = 0_v, T_v v, v \in V$ .

$$(1) \implies \pi_* f_*(x) = 0_{\pi(x)}$$

Notation:  $\zeta: X \rightarrow E$  is zero section;

$$\text{let } Z = \zeta(X) = \{(x, 0): x \in X\}.$$

Structural stability, plus (2) & (3), mean that

$$\tilde{f}_{0*} \circ \tilde{\nu}(v) = 0_{\tilde{\nu}(v)}, v \in V$$

$$\implies \tilde{f}_{0*} \circ \tilde{\nu}(V) \subset Z_x$$

$$\rightarrow \left\{ \begin{array}{l} \tilde{\xi}_0 \circ \kappa \circ \tilde{\gamma}(V) \subset Z_0 \\ \tilde{\xi}_C \circ \kappa \circ \tilde{\gamma}(V) \subset Z_C \end{array} \right\} \quad (4)$$

Assume

$$\left. \begin{array}{l} \text{(i)} \quad \xi_C \not\approx Z_C \\ \text{(ii)} \quad (\xi_C \circ \kappa)^{-1}(Z_C) \text{ is a submanifold of } K. \end{array} \right\} \quad (5)$$

By 'density' theorems,

$\exists \tilde{\xi}_0 \in N_0$  such that

$$\tilde{\xi}_0 \circ \kappa \not\approx Z_0 \quad (6a)$$

$$(\tilde{\xi}_0 \circ \kappa) \mid (\xi_C \circ \kappa)^{-1}(Z_C) \not\approx Z_0 \quad (6b)$$

(To justify this step in detail need

Lemma: let  $K \subset X$ ,  $Z \subset Y$  closed, embedded submanifolds.

Then

$$\{f: (f|K) \not\approx Z\}$$

is open and dense in  $C^\infty(X, Y)$ .

Proof (sketch): Start with  $g = f|K$ , in a canonical coordinate patch.

Give  $g$  a small, polynomial perturbation  $g \mapsto \tilde{g}$ , to achieve  $\tilde{g} \not\approx Z$ .

Smooth to get  $\tilde{f} = \tilde{g}$  on  $K$ ,  $\tilde{f} = f$  off a nbhd of  $K$ . Technique as in Golubitsky & Guillemin, p. 55, proof of Thom transversality theorem.)

Note that if (6a) holds and  $\tilde{\xi}_0 \circ \kappa(K) \cap Z_0 \neq \emptyset$ , as is assumed by structural stability, then necessarily  $n_0 \leq n - q$ , viz.  $q \leq m + n_C$ .

In (6b), write

$$K_C := (\xi_C \circ \kappa)^{-1}(Z_C)$$

$$\phi := \tilde{\xi}_0 \circ \kappa$$

Have

$$(D\phi)(T_x K_C) + T_{\phi(x)} Z_0 = T_{\phi(x)} E_0$$

$$\Rightarrow T_x K_C + (D\phi)^{-1} T_{\phi(x)} Z_0 = T_x K$$

$$\Rightarrow K_C \not\approx \phi^{-1}(Z_0)$$

i.e.

$$(\xi_C \circ \kappa)^{-1}(Z_C) \not\approx (\tilde{\xi}_0 \circ \kappa)^{-1}(Z_0) \quad (6c)$$

Introduce dimensions

$$\begin{array}{ccccccc} X & V & E_0 & E_C & K \\ n & m & n+n_0 & n+n_C & n-q \end{array}$$

$$n_0 := n - m - n_C$$

Let

$$\gamma_0 := \dim (\tilde{\xi}_0 \circ \kappa)^{-1}(Z_0)$$

$$\gamma_C := \dim (\xi_C \circ \kappa)^{-1}(Z_C)$$

Now

$$(6a) \Rightarrow \text{codim } (\tilde{\xi}_0 \circ \kappa)^{-1}(Z_0) = \text{codim } (Z_0)$$

$$\Rightarrow n - q - \gamma_0 = \dim (E_0) - \dim (Z_0)$$

$$= (n + n_0) - n$$

$$= n_0$$

$$\begin{aligned} \Rightarrow \gamma_0 &= n - q - n_0 \\ &= m + n_c - q \end{aligned} \quad (7)$$

$$\begin{aligned} (6c) \Rightarrow \dim [(\xi_c \circ \kappa)^{-1}(z_c) \cap (\tilde{\xi}_0 \circ \kappa)^{-1}(z_c)] \\ = \max \{0, \gamma_0 + \gamma_c - (n - q)\} \end{aligned} \quad (8)$$

$$(4) \Rightarrow \dim [\dots] \geq \dim \tilde{v}(V) = m \quad (9)$$

$$(8) \text{ \& } (9) \Rightarrow \gamma_0 + \gamma_c - n + q \geq m \quad (10)$$

$$(7) \text{ \& } (10) \Rightarrow \gamma_c \geq n - n_c \quad (11)$$

$$\text{But } \gamma_c \leq n - q \quad (12)$$

$$(11) \text{ \& } (12) \Rightarrow \text{MAIN RESULT: } n_c \leq q. \quad (13)$$

Interpretation:

The control section reduces to zero on a certain submanifold of  $K$ , namely  $K_c$ , of dimension  $\gamma_c$ , where  $\gamma_c \geq n - n_c$ . By (13) at least  $q$  'components' of control vanish on  $K_c$ , and this fact has the engineering interpretation that the inputs to at least  $q$  'integrators' in the control loop reduce to zero when the 'error' is zero. In other words, we have a version of the Internal Model Principle for nonlinear regulation against step inputs. Constructing a tubular nbhd of  $\kappa(K_c)$  in  $X$ , we may parametrize it as usual by the fibers, and the  $n - \gamma_c \geq q$  'components' of the fiber vectors provide a specific realization of the required feedback variables.

We can be more precise about  $\gamma_c$  if we

Assume:  $x(t, x_0) \rightarrow \tilde{v}(V)$  as  $t \rightarrow \infty$ , all  $x_0$ .

$$\text{Then } \tilde{\xi}_c^{-1}(z_c) = \tilde{v}(V)$$

$$\Rightarrow \tilde{\xi}_0^{-1}(z_0) \cap \tilde{\xi}_c^{-1}(z_c) = \tilde{v}(V)$$

$$\Rightarrow \gamma_0 + \gamma_c - n + q = m \text{ in (10)}$$

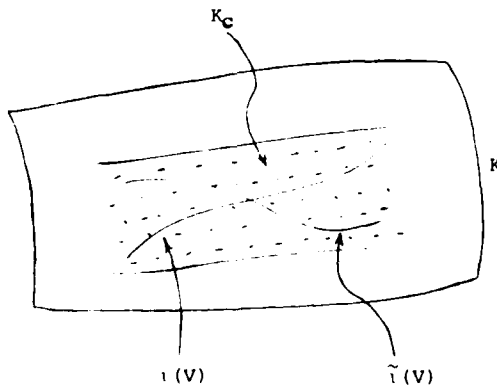
$$\Rightarrow n_c = q \text{ in (13).}$$

Recall  $K_c := (\xi_c \circ \kappa)^{-1}(z_c)$ .

$$\text{Then } \gamma_c = \dim(K_c)$$

$$= n - n_c = n - q = \dim(K)$$

Thus  $K_c$  is some closed submanifold of  $K$ , having the same dimension.



## 7. AN INTERNAL MODEL PRINCIPLE FOR CONTROLLED SEQUENTIAL MACHINES

### 7.1 Introduction

A long-standing idea in the control literature, and in several other disciplines, is that good control, on the part of an organism or artifact, depends on the formation of an internal model of the outside world. We shall discuss one way in which this 'internal model principle' can be formalized, in the context of controlled sequential machines. In conclusion we speculate on the implication for future control theories.

First, some citations from the literature.

1. O.J.M. Smith (1958):

"Feedback design techniques [can be] based on predictors which incorporate models of the mechanism of generation of the various signals being predicted".

2. C.R. Kelley (1968):

"Prediction and planning involve the operator's internal model of the control process and the variables in the environment that affect it ... Man's ability to interact with the environment depends on the accuracy of his internal model of that environment".

3. K. Oatley (1972):

"One fruitful idea is that the brain is a complex information-processing device that contains an internal model of the outside world...The brain, in other words, acts as a model in which neural processes symbolize the workings of the external world and thus allow us to predict the outcome of events and of our own actions".

[Credit to K. Craik (1967)].

4. James Albus & John Evans (1976):

"Efforts to equip a robot with modest intellectual capacities have occupied workers in the field of artificial intelligence for many years. In most instances the effort has required that the upper levels of control incorporate some internal data structures - a "world model", or knowledge frame - that can represent the state of the environment in a meaningful way. The robot must also be endowed with an adequate repertory of sensors and data-processing circuits for analysing the environment so that the robot can keep its internal knowledge frame up to date".

### 7.2 Abstract Internal Model Principle

System is abstract automaton

State transition map  $\alpha: X \rightarrow X$

Controller  $\gamma: X \rightarrow W: x \mapsto \gamma(x) = w$

$W$  is controller state set.

Good set  $K \subset X$

$K$  is where you would like to be.

{Typical situation:  $X = X_s \times X_c$

$K = K_s \times X_c, \quad K_s \subset X_s$

$\gamma: (x_s, x_c) \mapsto x_c$  }

Assume:  $\gamma(K) = W$

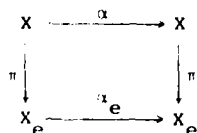
i.e. "Knowledge merely that  $x \in K$  yields no information about value of  $\gamma(x) \in W$ ."

Exosystem - "outside world as seen by controlled system". Assume

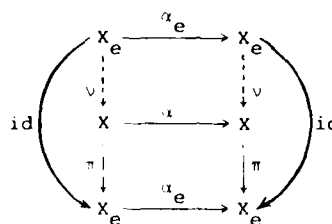
description incorporated in  $\alpha$ , viz. there is a subsystem  $X^+ \subset X$ ,

$X^+ \subset X, \quad \alpha(X^+) \subset X^+.$

$X^+$  could be determined as follows. Postulate that the exosystem is modeled by a pair  $(x_e, a_e)$  that drives the total system  $(X, \alpha)$  in accordance with the diagram



Here  $\pi$  is a surjection. Now assume that  $(X_e, \alpha_e)$  is coupled to  $(X, \alpha)$  in such a way that the total 'action' due to  $(X_e, \alpha_e)$  is uniquely determinable as an induced subsystem  $(X^+, \alpha^+)$ , viz. there exists a unique injection  $v: X_e \rightarrow X$  such that we have the diagram:



$$\begin{aligned}
 X^+ &:= v(X_e) \\
 \alpha^+ &:= \alpha|_{X^+}
 \end{aligned}$$

#### Regulation condition

$$X^+ \subset K$$

#### Detectability and observers

Consider time history  $\{x(t): t \in Z^+\}$ ,

$$x(0) = x_0 \in X$$

$$x(t+1) = \alpha(x(t)), \quad t \geq 0.$$

The resulting sequence of control states is

$$w(t, x_0) = \gamma(x(t)), \quad t \in Z^+$$

We would like to recover  $x_0$  from

$$\{w(t, x_0): t \in Z^+\}, \text{ for certain } x_0.$$

For this we need concept of observer.

#### Observers

Let  $\theta: X \rightarrow T$  be an 'output' map.

Write  $x \equiv x' \pmod{\theta} \iff \theta(x) = \theta(x')$ .

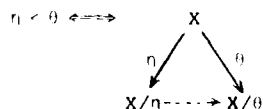
Identify  $\theta \sim$  equivalence relation on  $X$ .

Let  $E(X)$  be class of all equivalence relations on  $X$ .

If  $\theta \in E(X)$ , also identify  $\theta$  with projection  $X \rightarrow X/\theta$ .

Also,  $\theta \sim$  partition of  $X$ .

If  $\theta, \eta \in E(X)$ , say  $\eta < \theta$  ( $\eta$  is finer than  $\theta$ ) if each equivalence class of  $\eta$  is a subset of some equivalence class of  $\theta$ .



i.e.  $\theta$  factors through  $\eta$ .

Fact:  $<$  is a partial order on  $E(X)$  and induces a lattice structure.

Namely for any  $\eta, \theta \in E(X)$  there exists  $\phi := \eta \wedge \theta$  with the properties

$$\phi < \eta \quad \& \quad \phi < \theta$$

$$\& \quad (\forall \psi): \psi < \eta \quad \text{and} \quad \psi < \theta \implies \psi < \phi$$

Similarly there exists  $\omega := \eta \vee \theta$

(replace  $<$  by  $>$  in definition of  $\wedge$ )

Fact:  $E(X)$  is complete, i.e.

$$F \in E(X) \implies \sup F, \inf F \in E(X).$$

Note:  $0 := \inf E(X) = \{\{x\}: x \in X\}$

$$1 := \sup E(X) = \{X\}$$

Can now define:

Given  $(\theta, \alpha)$ ,  $\alpha: X \rightarrow X$ ,  $\theta \in E(X)$ , the corresponding observer is  $\omega \in E(X)$ ,

$$\omega := \sup \{\omega': \omega' \in E(X) \quad \& \quad \omega' < \theta \wedge (\omega' \circ \alpha)\}$$

(Here  $x \equiv x' \text{ mod } \omega' \circ \alpha \iff \alpha(x) = \alpha(x') \text{ mod } \omega'$ )

$\omega$  is the coarsest partition of  $X$  that is finer than  $\theta$  and is also a congruence for  $\alpha$ .

Can show:  $\omega = \inf \{\theta \circ \alpha^{i-1}, i \geq 1\}$ .

Say:  $(\theta, \alpha)$  is observable if  $\omega = 0$ .

Now let  $E \subset X$ ,  $\alpha(E) \subset E$ ,  $\alpha_E = \alpha|_E$  etc.

If  $\omega$  is observable for  $\alpha$  then  $\omega_E$  is observable for  $\alpha_E$ .

$E$  is detectable rel.  $(\theta, \alpha)$  if  $(\theta_E, \alpha_E)$  is observable.

Assume:  $X^+$  detectable rel.  $(\gamma, \alpha)$ .

### Feedback

Controller is actuated externally only when system state deviates from good set  $K$ .

Let  $x \in K$ .

Control state is  $\gamma(x) \in W$ .

Successor control state is  $\gamma \circ \alpha(x) \in W$ .

Assume:  $\gamma \circ \alpha(x)$  depends only on  $\gamma(x)$ .

$$\text{viz. } (\gamma \circ \alpha)_K = \gamma_K$$

### Summary of assumptions

$$X, \quad \alpha: X \rightarrow X \quad (1)$$

$$W, \quad \gamma: X \rightarrow W \quad (2)$$

$$K \subset X, \quad \gamma(K) = W \quad (3)$$

$$X^+ \subset K, \quad \alpha(X^+) \subset X^+ \quad (4)$$

$$X^+ \text{ detectable rel. } (\gamma, \alpha) \quad (5)$$

$$\gamma_K < (\gamma \circ \alpha)_K \quad (6)$$

$$\text{Put } \alpha^+ := \alpha|_{X^+}, \quad \gamma^+ = \gamma|_{X^+}$$

### Theorem

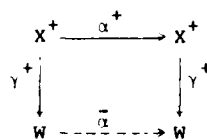
(i)  $\exists!$  map  $\bar{\alpha}: W \rightarrow W$  determined by

$$(\bar{\alpha} \circ \gamma)|_K = (\gamma \circ \alpha)|_K$$

(ii)  $\bar{\alpha} \circ \gamma^+ = \gamma^+ \circ \alpha^+$

(iii)  $\gamma^+$  is injective.

We have



### Proof

(i) Let  $w \in W$ .

(3)  $\Rightarrow \exists x \in K, \gamma(x) = w$ .

Let  $\bar{\alpha}(w) := \gamma \circ \alpha(x)$ .

OK? Let  $x' \in K, \gamma(x') = w$ .

$\Rightarrow x \equiv x' \pmod{\gamma}$

$\Rightarrow x \equiv x' \pmod{\gamma \circ \alpha}$ , by (6)

$\Rightarrow (\gamma \circ \alpha)x = (\gamma \circ \alpha)(x')$

(ii) Let  $x \in X^+$ . By (i),

$X^+ \cap K \Rightarrow \bar{\alpha} \circ \gamma(x) = \gamma \circ \alpha(x)$

$\alpha(x) = \alpha^+(x) \quad \& \quad \alpha(x) \in X^+ \Rightarrow \bar{\alpha} \circ \gamma^+(x) = \gamma^+ \circ \alpha^+(x)$ .

(iii) Let  $\omega = \text{observer for } (\gamma, \alpha)$ .

$\omega^+ := \omega \circ X^+$ , etc.

(5)  $\Rightarrow \omega^+ = \sup \{ \omega^1 : \omega^1 \in E(X^+) \quad \& \quad \omega^1 < \gamma^+ \wedge (\omega^1 \circ \alpha^+) \}$   
 $= 0$ .

But (ii)  $\Rightarrow \omega^+ < \gamma^+ \circ \alpha^+$ .

So  $\gamma^+ \circ \alpha^+ = 0$

i.e.  $\gamma^+$  is injective. QED

### 7.3 Conclusion

While the role of internal models now appears to be central in many processes of prediction and regulation, formal demonstrations of the existence of an internal model, and of how an internal model may be used (together with a stabilizer) in a feedback loop to achieve good control, are still few and somewhat specialized. However, such facts as we possess do point to the intriguing possibility of a 'general systems theory' considerably deeper and more substantial than exists under that title at the present time.

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## MULTIVARIABLE DESIGN : THE OPTIMIZATION OF APPROXIMATE INVERSES

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A new mathematical framework for the analysis of sensitivity in multi-input-multi-output linear feedback systems is proposed, based on the concepts of a multiplicative seminorm and an approximate inverse.

Many of the empirical results of "classical" control theory pertaining to the use of lead-lag networks can be deduced in this framework as solutions to well-posed mathematical optimization problems. Moreover, new classes of optimal filters for the reduction of sensitivity are introduced.

A definition of optimal sensitivity to plant uncertainty is established. The multiplicative property of seminorms is used to obtain the following principle: For any specified aposteriori accuracy, there is a maximum of apriori plant uncertainty that can be tolerated, and a minimum of identification that is required.

## INTRODUCTION

In this expository summary of Zames (1976-81) we are concerned with the effects of feedback on uncertainty, where uncertainty occurs either in the form of an additive disturbance  $d$  at the output of a linear plant  $P$ .

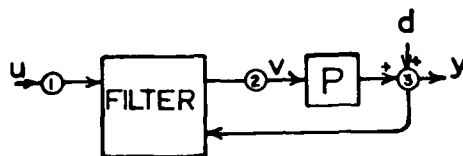


Fig. 1

or an additive perturbation in  $P$  representing "plant uncertainty". We approach this subject from the point of view of classical sensitivity theory, with the difference that feedbacks will not only reduce but actually optimize sensitivity in an appropriate sense.

The theory is developed at two levels of generality. At the higher level, a framework is sought in which the essence of the classical ideas can be captured. To this end, systems are represented by mappings belonging to a normed algebra. The object here is to obtain general answers to such questions as: How does the usefulness of feedback depend on plant invertibility? Are there measures of sensitivity to plant-uncertainty that are natural for optimization? How does plant uncertainty affect the possibility of designing a feedback scheme to reduce plant uncertainty?

At a more practical level, the theory is illustrated by simple examples of involving single-variable and multivariable frequency responses. The questions here are: Can the classical "lead-lag" controllers be derived from an optimization problem? How do rhp (right-half-plane) zeros restrict sensitivity? In multivariable systems without rhp zeros, can sensitivity be made arbitrarily small, and if so how?

A few observations might serve to motivate this re-examination of feedback theory. One way of attenuating disturbances is to introduce a filter of the W.H.K. (Wiener-Hopf-Kalman) type in the feedback path. Despite the unquestioned success of the W.H.K. and state-space approaches, the classical methods, which rely on lead-lag "compensators" to reduce sensitivity, have continued to dominate many areas of design. On and off there have been attempts to develop analogous methods for multivariable systems. However, the classical techniques have been difficult to pin down in a mathematical theory, partly because the purpose of compensation has not been clearly stated. One of our objectives is to formulate the compensation problem as the solution to a well-defined optimization problem.

Another motivating factor is the gradual realization that classical theory is not just an old-fashioned way of doing WHK, but is concerned with a different category of mathematical problems. In a typical WHK problem, the quadratic norm of the response to a disturbance  $d$  is minimized by a projection method; in a deterministic version, the power spectrum  $|d(j\omega)|$  is a single, known vector in, e.g., the space  $L_2(-\infty, \infty)$ ; in stochastic versions,  $d$  belongs to a single random process of known covariance properties. However, there are many practical problems in which  $d$  is unknown but belongs to a prescribed set of disturbances, or to a class of random processes whose covariances are uncertain but belong to a prescribed set. For example, in audio design,  $d$  is often one of a set of narrow-band signals in the 20-20K Hertz interval, as opposed to a single, wide-band signal in the same interval. Problems involving such disturbance sets are not tractable by W.H.K. or projection techniques. One objective here is to find a systematic approach to problems involving such sets of disturbances.

Another observation is that many problems of plant uncertainty can be stated easily in the classical theory, e.g. in terms of a tolerance-band on a frequency response, but are difficult to express in a linear-quadratic-state-space framework. One reason for this is that frequency-response descriptions and, more generally, input-output descriptions preserve the operations of system addition and multiplication, whereas state-space descriptions do not. Another reason is that the quadratic norm is hard to estimate for system products, whereas the induced norm (or "gain") that is implicit in the classical theory is easier to estimate. We would like to exploit these advantages in the study of plant uncertainty.

## Weighted Seminorms and Approximate Inverses

One way of defining the optimal sensitivity of a feedback system, and of addressing these issues is in terms of an induced norm of the sensitivity operator. However, it is shown Zames (1979) that the primary norm of an operator in a normed algebra is useless for this purpose. Perhaps that is why operator norm optimization has not been pursued extensively in the past.

Instead, we introduce an auxiliary "weighted" seminorm, which retains some of the multiplicative properties of the induced norm, but is amenable to optimization. Plant uncertainty is described in terms of belonging to a sphere in the weighted seminorm.

Approximate invertibility of the plant is one of the features which distinguishes control from, say, communication problems. We define the concept of an approximate inverse under a weighted seminorm, and show that sensitivity reduction is possible if and only if there is such an inverse.

## ALGEBRAS OF SYSTEMS

A feature of the input-output approach is that systems can be added, multiplied by each other or by scalars, and the sums or products obtained are still systems; i.e., causal input-output mappings from an algebra. The algebra of all such mappings under consideration will be denoted by  $\mathcal{A}$ . For example,  $\mathcal{A}$  can be an algebra of (possibly multivariable) causal frequency response mappings.  $I$  denotes the identity in  $\mathcal{A}$ .

Suppose that  $\mathcal{A}_s$  denotes the strictly causal systems in  $\mathcal{A}$ , i.e.,  $\mathcal{A}_s$  is a subalgebra of  $\mathcal{A}$  defined by the property: for any plant  $P \in \mathcal{A}_s$  and feedback  $F \in \mathcal{A}$ , the c.l. (closed-loop) operators  $(I + FP)^{-1}$  and  $(I + PF)^{-1}$  are well defined. Strict causality is a generalization of the physical property that (strictly proper) systems can not respond instantly to sudden inputs. Mathematically,  $\mathcal{A}_s$  is a radical of  $\mathcal{A}$ . However, all that concerns us here is that the inverses shown exist.

The subalgebra of stable systems in  $\mathcal{A}$  will be denoted by  $\mathcal{B}$ . Suppose that a norm,  $\|\cdot\|$ , is defined on  $\mathcal{B}$  and that  $\mathcal{B}$  is complete. An example of such a normed algebra is the (Hardy) space  $H^\infty$  of frequency response  $\hat{p}(s)$  analytic and bounded in the rhp  $\text{Re}(s) > 0$ , where the norm is the supremum of  $|\hat{p}(s)|$  over the rhp. The symbol  $\mathcal{B}_s$  will denote the subalgebra of strictly causal systems of  $\mathcal{B}$ .

## COMPENSATOR PARAMETRIZATION

A feedback system with strictly causal plant  $P$  and feedback  $F$  in  $\mathcal{A}_s$  can be represented by the matrix of 4 c.l. operators

$$\begin{bmatrix} (I + FP)^{-1} & P(I + PF)^{-1} \\ F(I + PF)^{-1} & (I + PF)^{-1} \end{bmatrix} \quad (1)$$

$(I + PF)^{-1}$  is the c.l. response to a disturbance at the plant output, and will be called the sensitivity operator. The initial objective is to make this sensitivity small in some appropriate sense, while keeping all 4 operators stable. Our first step will be to separate the desensitization problem from stabilization by a change of variable.

Initially, assume  $P$  stable. Let  $Q \triangleq F(I + PF)^{-1}$ . The matrix (1) can be expressed as

$$\begin{bmatrix} (I - QP) & P(I - QP) \\ Q(I - QP) & (I - PQ) \end{bmatrix} \quad (2)$$

It can be shown that (1-2) are stable if and only if  $Q$  is stable. In effect,  $Q \in \mathcal{B}$  is a parametrization of all feedbacks that preserve c.l. stability. By designing  $Q$  instead of  $F$  we replace a potentially unstable operator by a stable one and, more importantly, automatically guarantee closed-loop stability; no need to keep checking Nyquist's criterion, etc. The expression for sensitivity assumes the form  $(I - PQ)^{-1}$ , which will be convenient in studying the relation between sensitivity and invertibility of  $P$ .

If a plant  $P_0$  is unstable but is stabilized by a feedback  $F_0 \in \mathcal{B}_s$ , which makes  $P_0(I + F_0 P_0)^{-1}$  stable, then design can be separated into 2 stages: (1) stabilization by  $F_0$ ; (2) desensitization by  $Q$ . In other words, the pair of operators  $(F_0, Q)$  parametrizes the set of all stabilizing compensators, in a manner which separates design for say, low sensitivity, from stabilization.

## LIMITATIONS OF NORMS

For low sensitivity,  $Q$  must be designed to make  $(I - PQ)$  small. If  $P$  had a right inverse, the choice  $P = Q^{-1}$  would make sensitivity identically zero. However, it can be shown that strictly causal operators have no exact inverses in  $\mathcal{B}$  (just as strictly proper frequency responses have no strictly proper inverses). Instead we try to make  $Q$  an approximate right inverse of  $P$ , by making  $(I - PQ)$  small in some sense. Indeed, the various approaches to feedback theory can be classified according to the way in which they measure  $(I - PQ)$  and construct this approximate inverse. At this point we note two peculiarities (Zames, 1979).

(a)  $\|I - PQ\| \geq 1$ , i.e., it is impossible to make sensitivity small in the primary norm of a Banach algebra. Ex., for frequency responses,  $\hat{p}(j\omega)\hat{q}(j\omega) \rightarrow 0$  as  $|\omega| \rightarrow \infty$ , so sensitivity approaches 1. It is possible to make  $\|I - PQ\|$  small if  $(I - PQ)$  is restricted to a finite band of frequencies, or, more generally, an invariant subspace of  $\mathcal{B}$ . However,

(b) the infimum over all compensators  $Q \in \mathcal{B}$  of  $\|I - PQ\|$  restricted to an invariant subspace of  $\mathcal{B}$  is always either zero or one. For non-minimum phase systems, it can be shown that if sensitivity is made to approach zero over one subspace by choice of  $Q$ , then it approaches infinity over the complementary subspace. The

norm of  $(I - PQ)$  over an invariant subspace is therefore not a useful measure of sensitivity for optimization purposes.

Properties (a-b) delimit the peculiarities of the sensitivity reduction problem. In one form or another they were recognized in classical theory, and are perhaps the reason why sensitivity reduction was not posed as an optimization problem.

Our approach will be to employ an auxiliary norm or, more generally, seminorm (a nonzero element has nonzero norm, but may have zero seminorm) relative to which  $(I - PQ)$  can be made small.

#### WEIGHTED SEMINORMS

(We shall consider symmetric seminorms here. In the general theory, left and right seminorms are employed.)

Suppose that an auxiliary seminorm  $\|\cdot\|_W$  is defined on  $B$ , which will be called a weighted seminorm, and has the property that  $\|P\|_W \leq \|P\|$  for all  $P \in B$ , i.e.,  $\|\cdot\|$  "dominates"  $\|\cdot\|_W$ . Let  $\|\cdot\|_W$  be held fixed throughout.  $Q \in B$  is called an approximate right inverse of  $P \in B$  relative to  $\|\cdot\|_W$  if  $\|I - PQ\|_W \leq \|I\|_W$ . The measure of right singularity of  $P$  is

$$\mu(P) \triangleq \inf_{Q \in B} \|I - PQ\|_W$$

$\mu(P)$  is a number between 0 and 1 which measures the noninvertibility of  $P$  (and which, for frequency responses  $\hat{p}(j\omega)$ , usually depends on the rhp zero locations of  $\hat{p}$ .)

**THEOREM** For any plant  $P \in B$ , there is a sequence (of compensators)  $F_n \in B_s$ ,  $n = 1, 2, \dots$ , for which the sensitivities  $\|(I + PF_n)^{-1}\|$  approach  $\mu(P)$ , but no compensator can give a sensitivity less than  $\mu(P)$ .

i.e., the optimal sensitivity coincides with the measure of singularity. Feedback can reduce sensitivity iff  $P$  has an approximate right inverse.

In our framework, a disturbance attenuation filter of the WHK type can be described as follows: If the disturbance has a power spectrum  $\hat{\delta}(j\omega)$ , the weighted seminorm is the root-integral-squared,

$$\|I - PQ\|_{L_2} \triangleq \left( \int_{-\infty}^{\infty} |1 - \hat{p}(j\omega)\hat{q}(j\omega)|^2 \hat{\delta}(j\omega) d\omega \right)^{1/2}$$

and  $\mu(P)$  is the irreducible error.

Such quadratic norms present a difficulty in problems involving plant uncertainty, because they lack the multiplicative property  $\|AB\| \leq \|A\| \cdot \|B\|$  (needed to make  $B$  into a "normed-algebra"). If  $\Delta P$  is a plant perturbation resulting from, say, the identification of  $P$ , the closed-loop perturbation has the product form  $(I - PQ)\Delta P$ , and is difficult to estimate without a multiplicative property. Indeed, the product can become small even as its factors become large. Instead, we shall concentrate on

#### Multiplicative Seminorms

A weighted seminorm  $\|\cdot\|_W$  on  $B$  is called multiplicative (symmetric) iff it satisfies the following two inequalities for all  $A, B \in B$ .

$$\|AB\|_W \leq \|A\| \cdot \|B\|_W, \quad \|AB\| \leq \|A\|_W \cdot \|B\|$$

A simple example of a multiplicative seminorm on the space  $H^\infty$  of frequency response is obtained by taking a strictly proper weighting filter  $\hat{w} \in H^\infty$  with  $|w(s)| \leq 1$  in  $\text{Re}(s) \geq 0$ , and letting

$$\|P\|_W \triangleq \max |\hat{p}(j\omega)\hat{q}(j\omega)| \quad (3)$$

If  $\hat{p}(s)$  has rhp zeros  $z_i$ , then it is shown in Zames (1979) that  $\mu(P) \geq \hat{w}(z_i)$ ,  $i = 1, 2, \dots$ , so it is impossible to achieve small sensitivity if zeros are present in any heavily weighted part of the rhp.

#### Minimax Filters

For seminorms of the form (3), the optimization problem takes the form

$$\inf_{Q \in B_s} \sup_{\omega} |1 - \hat{p}(j\omega)\hat{q}(j\omega)| \hat{w}(j\omega)$$

This problem has explicit solutions which are detailed elsewhere. Let us merely give an example. Suppose that the plant  $\hat{p}(s) \triangleq (s - \beta)(s + \beta)^{-1}$  has one rhp zero, and  $\hat{w}(s)$  is the "lowpass" weighting  $k(s + k)^{-1}$ ,  $k > 0$ . Then  $\mu(P) = w(\beta)$ , and an optimal sequence of filters has the form  $\hat{q}_m(s) = c_m(s + \beta)(s + m)^{-1}(s + n_m)^{-2}$  where  $m, n_m = 1, 2, \dots$ , are integers and  $c_m \triangleq m^2 n_m (k + \beta)^{-1}$  is a constant. The filters  $\hat{q}_m(s)$  are "lead" networks whose bandwidth increases with  $m$ , and which approach a "proportional plus derivative" form as  $m \rightarrow \infty$ .

Our point here is not that minimax norms are better measures of cost than quadratic norms, but that minimax norms have the following advantages: (1) their multiplicative property makes them amenable to studies of plant uncertainty; (2) they enable us to design filters which perform well over sets\* of disturbances of imprecisely known power spectra; (3) the resulting optimal filters have a remarkable resemblance to the lead-lag filters of classical control theory.

\*There is a 1:1 correspondence between weighted seminorms and convex, balanced, absorbing sets of functions in a Banach space.

## MULTIVARIABLE SYSTEMS

Our objectives here are: firstly, to show again that optimal sensitivity is limited by the presence of heavily weighted rhp zeros; secondly, to show that sensitivity can be made arbitrarily small, by a sequence of filters of increasing bandwidth for which formulas are given, provided the plant matrix determinant does not vanish in the rhp, and does not decrease too quickly at high frequencies.

$H^{\infty N}$  denotes the algebra of  $N \times N$  matrices whose entries lie in  $H^{\infty}$ . The norm of any  $\hat{P}(\cdot) \in H^{\infty N}$  is the largest singular value of  $\hat{P}(j\omega)$ , denoted by  $\sup \sigma[\hat{P}(j\omega)]$ . (i.e., largest eigenvalue of  $\hat{P}(j\omega) \hat{P}^*(j\omega)$ ). The algebra of systems with  $H^{\infty N}$  frequency responses is denoted by  $H^{\infty N}$ .

A fixed, strictly proper, scalar weighting filter  $\hat{w}(\cdot) \in H^{\infty}$  is assumed to be given, and the multiplicative seminorm  $\|P\|_w = \sup \sigma[\hat{w}(j\omega)\hat{P}(j\omega)]$  is defined.

We again conclude that the measure of singularity satisfies  $\mu(P) \geq \hat{w}(s_0)$ ,  $s_0$  being any rhp zero of  $\det \hat{P}(s)$ .

**THEOREM** If  $\det \hat{P}(s) \neq 0$  for  $\text{Re}(s) \geq 0$ , and  $|\sigma[\hat{P}(s)]| \geq c|s|^{-k}$  for  $|s| > \rho$ , where  $c > 0$ ,  $\rho > 0$ , and  $k = 1, 2, \dots$  are constants, the sensitivity  $\|(I + PF)^{-1}\|_w$  can be made smaller than any  $\epsilon > 0$  by the compensator.  $\hat{Q}_n(s) = \hat{P}^{-1}(s) [n(s+n)^{-1}]^{k+1}$ ,  $n$  being a sufficiently large integer.

## DISTURBANCE ATTENUATION UNDER PLANT UNCERTAINTY

Two opposing tendencies can be found in most feedback systems. On the one hand, to the extent that feedback reduces sensitivity it reduces the need for plant identification. On the other hand, the less information is available about the plant, the less possible it is to select a feedback to reduce sensitivity. The balance between these tendencies establishes a maximum to the amount of tolerable plant uncertainty and, equivalently, a minimum to the amount of identification needed.

It can be argued that the search for such a minimum should be basic to the theory of adaptive systems. Actually, even the existence of such a minimum appears not to have been stated, perhaps because plant uncertainty is so difficult to study in the WHK framework in the absence of the multiplicative properties, and because there is no notion of optimality in the classical setup.

Here, we would like to take a step in the direction of articulating these issues, by defining the trade-off between minimal sensitivity and plant uncertainty and deducing its simpler properties. Sensitivity to disturbances will be considered in this section, and to plant uncertainty in the next.

Uncertainty in a plant  $P \in B_s$  can be described by stating  $P$  is an unknown element of some prescribed "ball" of plants, denoted by  $b(P_1, \delta)$ , centred at some nominal plant  $P_1$ , and of radius  $\delta > 0$ , i.e.,  $\|P_1 - P\| \leq \delta$ . For example, for a frequency-response  $\hat{P}(j\omega)$  this means that  $\hat{P}(j\omega)$  lies in a band of radius  $\delta$  centred around  $\hat{P}_1(j\omega)$ . This description of uncertainty may be cruder than a probabilistic one, but has the pragmatic advantage of being tractable under the nonlinear feedback transformation.

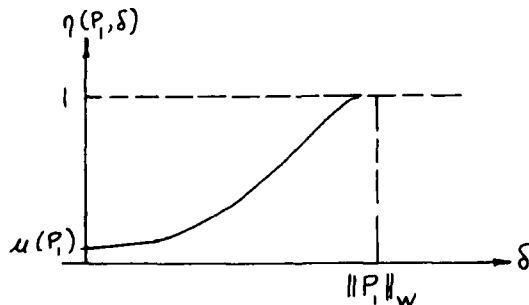
Let  $\|\cdot\|_w$  be a given multiplicative seminorm. If the plant is in the ball  $b(P_1, \delta)$ , we define the optimal sensitivity under plant uncertainty to be

$$\eta(P_1, \delta) = \inf_F \sup_{P \in b(P_1, \delta)} \|(I + PF)^{-1}\|_w$$

the inf being over all  $F \in B_s$  that maintain c.l. stability; i.e.,  $\eta$  is the optimized worst-case sensitivity. We obtain the following result.

**THEOREM** For any nominal plant  $P_1 \in B_s$  and radius of uncertainty  $\delta > 0$ , the optimal sensitivity  $\eta(P_1, \delta)$  is a monotone nondecreasing function of  $\delta$ , which approaches the singularity measure  $\mu(P_1)$  as  $\delta \rightarrow 0$ , and equals  $\|I\|_w$  for  $\delta = \|P_1\|_w$ .

In other words, the less information we have about  $P$ , the less we are able to design a feedback to attenuate disturbances.



# FILTERING OF PLANT UNCERTAINTY: INVARIANT SCHEMES

Finally, we turn to the problem of a plant  $P$  in a ball  $b(P_1, \delta)$  of uncertainty of radius  $\delta > 0$  centred at a nominal value  $P_1$ , and consider feedbacks that shrink the uncertainty. Any linear feedback arrangement for a plant that has only one (possibly multivariable) accessible input can be parameterized by two operators  $(F, R)$  as shown in the following flow-graph.

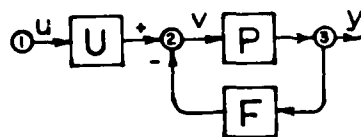


Fig. 2

Of course, uncertainty can be reduced to zero by disconnecting the input  $u$  from the system, but then  $P_1$  is also transformed into  $ze \cdot 0$ . Clearly, the problem is trivial unless there is a normalization or constraint on the control law that transforms  $P_1$  into a closed-loop system.

We would prefer as far as possible to separate the reduction of uncertainty from the transformation of  $P_1$ , and therefore seek a definition of uncertainty which is independent of the eventual closed-loop system.

If  $P_1$  were a real number, uncertainty could be normalized by specifying it as a percentage of the nominal value. This possibility is not open for noninvertible plants. Instead, we achieve a normalized definition of uncertainty by employing the device of a "plant-invariant scheme": i.e., a scheme which leaves the nominal plant invariant but shrinks the radius of uncertainty. Such a scheme is shown to be always realizable in the form of the following "model reference scheme":

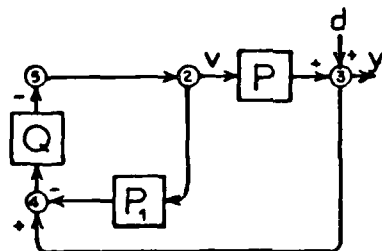


Fig. 3

This device also enables us to separate the design process into two consecutive stages: (1) reduction of uncertainty, and (2) transformation of the nominal plant into a nominal c.l. system. (cf. the separation into estimation and control stages in Kalman filtering.)

Suppose, then, that the feedback scheme is (normalized by being) plant invariant, and for simplicity here say that  $R = 0$ . Let  $\|\cdot\|_W$  be a given multiplicative seminorm.

For a plant  $P$  in a ball of uncertainty  $b(P_1, \delta)$  the optimal sensitivity to plant perturbations is defined to be

$$v(P_1, \delta) = \frac{1}{\delta} \inf_F \sup_{P \in b(P_1, \delta)} \|P(I + FP)^{-1} - P_1(I + FP_1)^{-1}\|_W$$

**THEOREM** For any nominal plant  $P_1$  in  $B_S$ , the infimal sensitivity to plant perturbations  $v(P_1, \delta)$  is a monotone nondecreasing function of  $\delta > 0$ .  $v(P_1, \delta)$  satisfies the upper bound conditions  $v(P_1, \delta) \leq \eta(P_1, \delta)$  and  $\lim_{\delta \rightarrow 0} v(P_1, \delta) \leq \mu(P_1)$ .

Again, there is a tradeoff between uncertainty about the plant and sensitivity reduction achievable by feedback.

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## FEEDBACK AND MINIMAX SENSITIVITY

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## ABSTRACT

In this paper, we look for feedbacks that minimize the sensitivity function of a linear, single-variable feedback system represented by its frequency responses. Sensitivity is measured in a weighted  $H^\infty$  norm.

In an earlier paper, Zames proposed an approach to feedback design involving the measurement of sensitivity by "multiplicative seminorms", which have certain advantages over the widely used quadratic norm in problems where there is plant uncertainty, or where signal power-spectra are not fixed out before to sets. The problem was studied in a general setting, and some  $H^\infty$  examples were solved.

Here, a detailed study of the single-variable case is undertaken. The results are extended to unstable plants, and explicit formulas for the general situation of a finite number of RHP plant zeros or poles are provided. The  $Q$  or "approximate-inverse" parametrization of feedbacks that maintain closed-loop stability is extended to the case of unstable plants. The  $H^\infty$  and Wiener-Hopf approaches are compared.

3) Problems involving plant perturbations or uncertainty are most naturally studied under norms with multiplicative properties of the type  $\|PQ\| \leq \|P\| \cdot \|Q\|$ , which  $H^\infty$  norm have but quadratic norms do not, and which allow the norm of a product of transfer functions to be related to the norms of the factors.

In Zames [1] some of the main features of our approach were outlined in a general setting, and examples were solved involving multivariable systems without RHP (right half-plane) zeros, and single variable systems with unstable RHP zeros, with emphasis on plants that are stable after feedback stabilization. Here, we undertake a detailed study of the single-variable case in the presence of plant uncertainty, extending the results to unstable plants, and provide explicit formulas for the general case of a finite number of RHP plant zeros or poles.

In [1] the feedbacks that maintain closed-loop stability were parametrized by a stable system-variable  $Q$ , which was interpreted as an approximate inverse. Here, we extend the parametrization to unstable plants, and interpret it as a constrained approximate inverse.

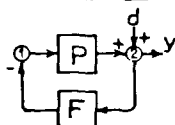
The  $H^\infty$  and Wiener-Hopf approaches are compared in an example.

Our results are an application of the theory of optimal interpolation developed early in the century by Cartheodory, Schur, Pick, Nevanlinna and others (see Walsh [2, Chapter 1-4]). An exposition of optimal  $H^\infty$  interpolation, involving the functional-analytic concept of duality, was developed several decades later by Rogosinski and Shapiro [3], and is outlined in Duren [4].

The  $Q$ -parametrization is related to the well known results of Youla-Bongiorno-Jahr [6], and the interesting generalizations recently obtained by Desoer-Liu-Murray-Sacks [7], for coprime factorable plants and feedbacks. The  $Q$ -parametrization does not preassume coprimeness. It is convenient here because it is affine in  $Q$ , and displays the relationship between sensitivity and invertibility.

Safonov-Athans [8] and Stein-Doyle [9] have employed maximum sensitivity constraints in quadratic optimization problems, and our motivation here overlaps theirs.

## 1. INTRODUCTION



(This is a summary, with proofs omitted, of a report obtainable from the authors.)

In this paper we study the linear servo problem, and seek a feedback  $F$  (see Fig. 1) to minimize the sensitivity to output disturbances  $d$ , at the output  $y$  of a linear single-variable plant  $P$  characterized by its frequency response. Sensitivity is measured in a weighted  $H^\infty$  (operator) norm.

There are several reasons for our interest in weighted  $H^\infty$  optimization.

1) Problems involving variable power spectra are not usually tractable by quadratic optimization methods, but are tractable by  $H^\infty$  methods.

2)  $H^\infty$  methods yield filters with the "broadband" quality that often distinguishes classical frequency response from IQG designs, and offer the possibility of a clearer mathematical basis for such ad hoc classical procedures as lead-lag compensation.

## II. PROBLEM FORMULATION

The feedback system of Fig. 1 consists of a plant and feedback "compensator", whose input-output behavior is represented by the frequency responses  $P(s)$  and  $F(s)$  respectively. The CL(closed-loop) behavior of this system is represented by  $2 \times 2$  matrix of frequency responses

$$[K_{ij}]_{i,j=1}^2 = \begin{bmatrix} (1 + PF)^{-1} & P(1 + PF)^{-1} \\ F(1 + PF)^{-1} & (1 + PF)^{-1} \end{bmatrix} \quad (2.1)$$

In particular, the response of the output  $y$  to a disturbance  $d$  is  $K_{22} = (1 + PF)^{-1}$ . This term appears in most expressions for sensitivity, whether to additive disturbances such as  $d$ , or to perturbations in the plant  $P$ . We shall call  $K_{22}$  the sensitivity function\*. Our objective will be to optimize certain measures of this sensitivity function while keeping the  $K_{11}$  stable.

A frequency response, for our purposes, is any function  $G(s)$  of a complex variable  $s = \sigma + j\omega$ , which is meromorphic in the open RHP(right half-plane)  $\text{Re}(s) > 0$ , i.e., is analytic in  $\text{Re}(s) > 0$ , except, perhaps, at a countable number of poles.

A frequency response  $G(s)$  is proper iff the limit  $\lim_{\omega \rightarrow \infty} \sup_{\sigma \geq 0} |G(s)|$  is finite, and strictly proper iff the limit is zero.  $G(s)$  is stable iff it is bounded, i.e.,  $|G(s)| \leq \text{const.}$ , and analytic in  $\text{Re}(s) > 0$ .

If  $G_1$  and  $G_2$  are stable frequency responses and  $c$  is any complex constant, then the sum  $G_1 + G_2$  and products  $cG_1$  and  $cG_2$  are all stable frequency responses, i.e., stable frequency responses form an algebra over the field of complex numbers. The norm of any stable frequency response  $G$  is defined to be  $\|G\| = \sup_{\sigma \geq 0} |G(s)|$ . Under this norm, the algebra of stable frequency responses is called the Hardy space of order  $\infty$ , and denoted by  $H^\infty$ . It can be shown that the  $H^\infty$  norm can be found from the magnitude  $G(j\omega)$  along the  $j\omega$ -axis, i.e.,  $\|G\| = \sup_{\omega} |G(j\omega)|$ . The strictly proper frequency responses in  $H^\infty$  form a subalgebra denoted by  $H_0^\infty$ .

It will be assumed throughout that  $P(s)$  and  $F(s)$  are frequency responses such that  $P(s)F(s)$  is not identically equal to  $-1$  (which will certainly be true if  $P(s)$  and  $F(s)$  are strictly proper). This assumption ensures that all the inverses in (2.1) are well-defined frequency responses. Furthermore, it will be assumed that  $P(s)$  is the sum of an  $H^\infty$  function and a proper rational function (so that  $P(s)$  is proper) and that  $P(s)$  has at most a finite number of poles and zeros in  $\text{Re}(s) \geq 0$ .

### 2A. Fractional Transformation

We shall employ a transformation which serves to parametrize the feedbacks that keep the CL system stable, and thereby ensures a stable design. This transformation was introduced in [1] for stable plants, and is extended here to unstable plants. Let

$Q$  be a new variable defined by the fractional transformation

$$Q = F(1 + PF)^{-1} \quad (2.2a)$$

The hypothesis that  $P(s)F(s) \neq -1$  ensures that (2.2a) is well-defined. Moreover,  $P(s)Q(s) \neq 1$ , and  $F(s)$  can be recovered from  $Q(s)$  by the reverse transformation

$$F = Q(1 - PQ)^{-1} \quad (2.2b)$$

and vice versa. We shall view  $Q(s)$  as a representation\* of the feedback  $F(s)$ , which is useful for design purposes. In terms of  $Q$ , (2.1) assumes the form

$$[K_{ij}] = \begin{bmatrix} (1 - QP) & P(1 - QP) \\ Q & (1 - PQ) \end{bmatrix} \quad (2.3)$$

If the matrix (2.3) is stable, then  $Q$  is stable. Conversely, any choice of stable  $Q$  makes (2.3) stable provided that  $W$  satisfies the following constraints. C1:  $PQ$  and  $(1 - PQ)P$  are both stable. By using  $Q$ , subject to these constraints, as a design variable instead of  $F$ , we replace a potentially unstable design variable by a stable one and, more importantly, automatically ensure closed-loop stability, without further recourse to stability tests.

If  $P$  is stable to begin with then C1 is satisfied automatically. Otherwise, C1 is a condition for  $Q$  to represent a stabilizing feedback for  $P$ . For stable  $Q$ , C1 is equivalent to the condition that C2: At any pole  $a$  of  $P(s)$  of order  $m$  in  $\text{Re}(s) \geq 0$ ,  $(1 - P(s)Q(s))$  must have  $m$  or more zeros.

That C1 implies C2 is clear. Conversely suppose that C2 is true. Then, as  $Q$  is bounded in  $\text{Re}(s) > 0$ , neither  $(1 - PQ)P$  nor  $(1 - PQ)$  can have any poles in  $\text{Re}(s) \geq 0$ , except possibly at the poles of  $P$ , where in fact both are now finite. Both  $(1 - PQ)P$  and  $(1 - PQ)$  must therefore be bounded and analytic in  $\text{Re}(s) > 0$ , and therefore stable, i.e., C1 is true.

Observe that under C2,  $Q$  has precisely  $m$  zeros at  $a$ .

### 2B. Behavior near $\infty$ ; Weightings

Some notation for limiting the growth or decay of frequency responses as  $s \rightarrow \infty$  will be adopted.

Notation A frequency response  $G$  is of inferior order  $O(-l)$  at  $\infty$ ,  $l \geq 0$ , iff for some  $R > 0$ ,

$$|G(s)| \leq \text{const}|s|^{-l} \quad (2.4)$$

for  $|s| > R$ ,  $\text{Re}(s) \geq 0$ . The statement that  $G$  is in  $(1 + s)^l H^\infty$  means that  $(1 + s)^{-l} G(s)$  is in  $H^\infty$ .

### 2C. The Problem

The main objective of this paper is to solve the following problems: Suppose that the plant  $P(s)$  and a weighting  $W(s)$  are fixed, proper frequency responses, not identically equal to zero, and

- 1)  $P(s)$  is the sum of an  $H^\infty$  function continuous

\*Observe that as  $P(s)$  is proper, (2.2a-b) map any strictly proper  $F(s)$  into a strictly proper  $Q(s)$ , and vice versa.



in  $\text{Re}(s) \geq 0$ , and a proper rational function;  $P(s)$  has  $q$  poles,  $a_1, \dots, a_q$ , and  $r$  zeros,  $b_1, \dots, b_r$ , in  $\text{Re}(s) > 0$ , ( $q, r$ , finite), each enumerated according to its multiplicity; no poles or zeros on the  $j\omega$ -axis; and  $P(s)$  has inferior order  $O(-k)$  at  $\infty$ .

2)  $W(s)$  is in  $H^\infty$ , is continuous in  $\text{Re}(s) \geq 0$ , has norm  $\|W\| \leq 1$ , has no zeros in  $\text{Re}(s) \geq 0$ , and is of inferior order  $O(-k)$  at  $\infty$ .

**Problem 1:** Find a strictly proper feedback frequency response  $F(s)$  to minimize

$$\|W(1 + PF)^{-1}\| \quad (2.5)$$

subject to the constraint that the CL matrix (2.1) is stable. As the value of (2.5) is identical to that of

$$\|W(1 - PQ)\| \quad (2.6)$$

an equivalent problem, which is easier to solve because (2.6) is affine in  $Q$  whereas (2.5) is non-linear in  $F$ , is: **Problem 2.** Find a stable strictly proper  $Q$  in  $H_0$  to minimize (2.6), subject to the constraint that  $Q$  stabilize the CL system, i.e., that conditions C1 or C2 hold.

The expression (2.5-2.6) will be called the sensitivity. The optimal sensitivity, denoted by  $\mu(P)$ , is defined by either one of the equivalent expressions

$$\mu(P) = \inf_F \|W(1 + PF)^{-1}\| \quad (2.7)$$

$$= \inf_Q \|W(1 - PQ)\| \quad (2.8)$$

where the infima are taken over all strictly proper frequency responses  $F$ , or over all  $Q$  in  $H_0$  satisfying the constraints C1 or C2.

#### Remarks

$W$  generates an auxiliary norm  $\|\cdot\|_W$  on the algebra of stable frequency responses, defined by  $\|G\|_W = \|WG\|$ , and our object therefore is to minimize this auxiliary norm of the sensitivity function,  $\|(1 + PF)^{-1}\|_W$ . The auxiliary norm has the "multiplicative" properties,

$$\|G_1 G_2\|_W \leq \|G_1\|_W \|G_2\|_W, \quad \|G_1 G_2\|_W \leq \|G_1\|_W \|G_2\| \quad (2.9)$$

It was pointed out in [1] that inequalities (2.9) are important in the study of plant perturbations or plant uncertainty, but are not valid for the quadratic norms used in LQG methods. One of our main reasons for pursuing the presented weighted- $H^\infty$  approach is that it can be extended to problems of plant uncertainty, along the lines of [1], although plant uncertainty will not be considered here.

#### 2E. Relaxation of Propriety

The optimization problem is simplified if the propriety constraints on  $Q$  and the other CL responses of the matrix (2.2) are relaxed. It will be convenient initially to solve the simpler problem (in Sect. 4), and later in (Sec. 5) to modify the improper solution obtained for  $Q$ , by inserting some high frequency attenuation, and so obtain a family of strictly proper solutions to the original problem. The relaxed problem is defined as follows.

Let  $H^B$  denote the algebra of functions  $G(s)$  which are analytic in  $\text{Re}(s) > 0$  and bounded on every half-disc,  $|s| < R$ ,  $\text{Re}(s) > 0$ , of finite radius  $R > 0$ .

**Problem 2<sup>0</sup>:** Find a (not necessarily proper)  $Q$  in  $H^B$  to minimize (2.6) subject to the constraint that

$$\text{C3 } (1 - PQ)P \text{ is in } H^B. \quad (\text{Note: } Q \in H^B, (1 - PQ) \in H^B \Rightarrow PQ \in H^B.)$$

This problem is divorced from any consideration of  $F$ .

### III. ALLOWABLE SENSITIVITIES AND FEEDBACKS: INTERPOLATION CONSTRAINTS

The optimization problems 2 and 2<sup>0</sup> will be transformed here into optimal interpolation problems. In particular, the constraints C2-C3 on  $Q$  will be expressed as interpolation constraints.

Suppose that  $P$  and  $W$  are fixed as in Sect. 2C and  $Q(s)$  is in  $H^B$ . We shall wish to characterize those  $Q(s)$  for which  $P(s)Q(s)$  is in  $H^B$ , and observe that there are three equivalent ways of characterizing  $Q(s)$ :

- 1)  $Q(s)$  has zeros at the poles of  $P(s)$  in  $\text{Re}(s) > 0$ , taking into account their multiplicities;
- 2)  $Q(s)$  satisfies the interpolation constraints  $Q(s) = 0, dQ/ds = 0, \dots, (d/ds)^{m-1} Q = 0$  at each distinct pole of  $P(s)$  of multiplicity  $m$  in  $\text{Re}(s) > 0$ ;

- 3) the ratio  $Q(s)/B_P(s)$  is in  $H^B$ , where

$$B_P(s) = \prod_{i=1}^q \frac{a_i - s}{a_i + s} \quad (3.2)$$

(recall that  $a_1, \dots, a_q$  are the  $\text{Re}(s) > 0$  poles of  $P(s)$ .)

A rational function of the form (3.2) is called a Blaschke product. In Engineering terminology, a frequency response is called allpass if its magnitude is constant a.e. on the  $j\omega$ -axis. As  $B_P(j\omega) = 1$ , Blaschke products are allpass.

#### Constraints induced by RHP Plant Poles

A (possibly improper) frequency response  $Q(s)$  in  $H^B$  satisfies the condition C3 for the CL system to be in  $H^B$  iff  $\{1 - P(s)Q(s)\}$  has zeros at the RHP poles of  $P(s)$ , taking into account their multiplicities. There are, again, several equivalent ways of expressing C3, namely

- 1)  $Q$  satisfies the interpolation constraints
 
$$\begin{aligned} \{1 - P(a)Q(a)\} &= 0 \\ &\vdots \\ (d/ds)^{m-1} [1 - PQ] &= 0 \text{ at } s = a \end{aligned} \quad (3.3a)$$

at each RHP pole  $a$  of  $P(s)$  of multiplicity  $m$ .

- 2) A compact way of expressing (3.3a) is to say that  $B_P$  divides  $(1 - PQ)$  in  $H^B$ , i.e.,

$$(1 - PQ)B_P^{-1} \in H^B \quad (3.3b)$$

or that

$$3) \quad W(1 - PQ)B_p^{-1} \in H^B \quad (3.3c)$$

(3.3c) is equivalent to (3.3b) because  $W$  and  $W^{-1}$  are in  $H^B$ , the latter because  $W(s)$  has no zeros in  $\text{Re}(s) \geq 0$ .

The conditions for stable  $Q(s)$  to stabilize  $P(s)$ , namely C1-C2, are equivalent to the requirement that the expression in (3.3b) be in  $H^\infty$  or that (3.3a) hold.

3A1. Prop. Suppose there exists a function  $Q_0(s)$  in  $H_0^\infty$  which stabilizes  $P(s)$ ; then

a)  $Q_0(s)$  can be expressed in the form  $Q_0(s) = \frac{Q_{0A}(s)B_p(s)}{P(s)}$ , where  $Q_{0A}$  is in  $H_0^\infty$ ;

b) The functions  $Q$  in  $H_0^\infty$  which stabilize  $P(s)$  are those and only those which have the form

$$Q = B_p Q_{0A} + B_p^2 Q_1 \quad (3.4)$$

where  $Q_1$  is in  $H_0^\infty$ .

c) More generally, the functions  $Q(s)$  in  $H^B$  for which  $(1 - PQ)B_p^{-1}$  is in  $H^B$  (or  $H^\infty$ ) are those and only those which can be expressed in the form (3.4), with  $Q_1(s)$  in  $H^B$  (and such that  $P(s)Q(s)$  is proper).

Remark There is a connection between Proposition 3A1 and Theorem 3 of [7], which is elaborated in the original report.

### 3B. Constraints Added by RHP Plant Zeros

We are interested in optimizing the weighted sensitivity function  $W(1 - PQ)$  which will be denoted by  $X$  and is related to  $Q$  by the equations

$$X = W(1 - PQ) \quad (3.5)$$

$$Q = P^{-1}(1 - W^{-1}X) \quad (3.6)$$

Let  $\Omega$  denote the set of those  $Q(s)$  in  $H^B$  that satisfy the pole constraints (3.3) and for which  $WPQ$  is proper. If  $Q$  is in  $\Omega$  and  $X$  is given by (3.5), then it follows from (3.3) and the propriety of  $WPQ$  that both  $X$  and  $XB_p^{-1}$  are in  $H^\infty$ . From (3.5) it is clear that  $X$  must satisfy the following plant-zero interpolation constraints;

$$\begin{aligned} X(b) &= W(b) \\ &\vdots \\ (d/ds)^{m-1} X &= (d/ds)^{m-1} W \text{ at } s = b \end{aligned} \quad (3.7)$$

at each distinct plant zero  $b$  of multiplicity  $m$ .

### Remarks on Stabilization and Desensitization vs. Inversion

To achieve a small sensitivity  $W(1 - PQ)$ ,  $PQ$  must be close to 1, i.e.,  $Q$  must act as an approximate inverse of  $P$  subject to the plant-pole constraints (3.3). Now (3.3a) means that only those approximate inverses are allowed which are exact at the RHP poles of  $P(s)$ . Equivalently, sensitivity is made small subject to the constraint that it is zero at the plant poles. From this viewpoint, the problems of sensitivity reduction and stabilization by feedback are related, and the latter can be viewed

as an extreme case of the former.

The minimal sensitivity  $\mu(P)$  can be interpreted as a measure of the constrained singularity of  $P$ .

### IV IMPROPER MINIMIZATION

We turn now to the optimization problem  $2^0$ , of finding the minimum

$$\mu(P) = \inf \{ \|X\| : X = W(1 - PQ) \} \quad (4.1)$$

under the relaxed constraint that  $Q(s)$  is a possibly improper function in  $H^B$  which satisfies the plant-pole constraints (3.3). (Equivalently,  $Q$  is in  $\Omega$ ). Here,  $P(s)$  and  $W(s)$  are the fixed functions defined in Sect. II, and it will be assumed that  $P(s)$  is either strictly proper or has at least one RHP zero.

$X$  can be minimized first, subject to (3.7), and the optimal  $Q$  calculated second by (3.6). Furthermore, as  $\|X\| = \|XB_p^{-1}\|$ , the minimization of  $X$  can be accomplished by minimizing  $XB_p^{-1}$  and multiplying the result by  $B_p$ .

Lemma 1 (a) There is a unique weighted sensitivity function  $X \triangleq W(1 - PQ)$  in  $H$  of minimum norm. The necessary and sufficient conditions for  $X$  to be that function are that  $X$  satisfy the interpolation constraints (3.7), and have the (allpass) form

$$\tilde{X}(s) = D \prod_{i=1}^m \frac{c_i - s}{c_i + s} \prod_{j=1}^q \frac{a_j - s}{a_j + s} \quad (4.2)$$

in which  $\text{Re}(c_i) \geq 0$ ,  $D$  is a constant satisfying  $|D| = \|\tilde{X}\|$ , and  $m=r-1$ .

(b) The optimal  $\tilde{Q}$  is  $\tilde{Q} = P^{-1}(1 - W^{-1}\tilde{X})$ , and belongs to  $(1 + s)^{k+l}H^\infty$ .

(c) If the plant  $P(s)$  and weighting  $W(s)$  have conjugate symmetry, then so do  $\tilde{X}$  and  $\tilde{Q}$ , and the coefficients in (4.2) are real or occur in conjugate pairs.

### V. MINIMIZATION OVER STRICTLY PROPER $Q$

Suppose that  $P(s)$  and  $W(s)$  are fixed as in Sect 2C, but will be assumed to have conjugate symmetry.  $\tilde{Q}(s)$  and  $\tilde{X}(s)$  are the extrema specified in Lemma 1, and  $Q_0(s)$  is any function in  $H_0^\infty$  representing a strictly proper feedback that stabilizes  $P(s)$ . Let  $Q_n(s)$ ,  $n = 1, 2, \dots$ , be defined in one of two ways:

Case 1. If  $W(s)$  is strictly proper,

$$\tilde{Q}_n(s) \triangleq Q_0(s) + (\tilde{Q} - Q_0)(s) [n(s+n)^{-1}]^{k+l+1} \quad (5.1)$$

Case 2. If  $W(s)$  is proper, and

$\lim_{|s| \rightarrow \infty, \text{Re}(s) \geq 0} |W(s)| \triangleq |W(\infty)| \leq |\tilde{X}(j\omega)|$ , then

$$\tilde{X}_{1n}(s) \triangleq \tilde{X}(s) [1 - B_z(s) \frac{\lambda s}{n+s}], \quad \lambda = \frac{\tilde{X}(\infty) - W(\infty)}{B_z(\infty) \tilde{X}(\infty)} \quad (5.2a)$$

$$\tilde{Q}_{1n}(s) \triangleq P^{-1}(s) [1 - \tilde{X}_{1n}(s) W^{-1}(s)] \quad (5.2b)$$

$$\tilde{Q}_n(s) \Delta \tilde{Q}_0(s) + [\tilde{Q}_{1n}(s)] [m_n(s+m_n)^{-1}]^{k+l} \quad (5.2c)$$

where, for any  $n > 0$ ,  $m_n > 0$  is a sufficiently large integer.

In either case, the corresponding sequence of feedbacks is given by the fractional transformation

$$\tilde{F}_n(s) = \tilde{Q}_n(s) [1 - P(s)\tilde{Q}_n(s)]^{-1} \quad (5.3)$$

If the numerator and denominator of (5.3) are divided by  $B_p^{-1}(s)$ , (5.3) can also be written as

$$\tilde{F}_n(s) = \tilde{Q}_n(s) B_p^{-1}(s) \{ [1 - P(s)\tilde{Q}_n(s)] B_p^{-1}(s) \}^{-1} \quad (5.4)$$

Theorem 1 The functions  $\tilde{Q}_n(s)$  defined by (5.1) in Case 1, or (5.2) in Case 2, are (strictly proper) in  $H_0^\infty$ , stabilize  $P(s)$ , and are optimal in the sense of producing sensitivities approaching the infimal value  $\mu(P)$ , i.e.,

$$\lim_{n \rightarrow \infty} \|W(1 - P\tilde{Q}_n)\| = \mu(P) \quad (5.5)$$

Similarly the feedbacks  $\tilde{F}_n(s)$  are in  $H_0^\infty$  and optimal, i.e.,

$$\lim_{n \rightarrow \infty} \|W(1 + \tilde{F}_n P)^{-1}\| = \mu(P) \quad (5.6)$$

Moreover,  $\mu(P) = \|\tilde{X}\|$ .

#### 5A. Bounds on the Minimal Sensitivity

$$\text{5A1 Prop. a) } \mu(P) \geq \max_i |W(b_i) B_p^{-1}(b_i)| \quad (5.7)$$

$$> \max_i |W(b_i)| \quad (5.8)$$

assuming  $P$  is unstable, i.e.,  $B_p \notin 1$ , in (5.8).

b) If  $P_1$  and  $P_2$  are plants with identical RHP zeros,  $P_1$  is stable, and  $P_2$  has poles in the RHP, then  $\mu(P_2) > \mu(P_1)$ .

Remark: It follows from (5.8-5.7) that small sensitivity can not be achieved if there are zeros in any heavily weighted part of the RHP, or poles anywhere near these zeros (since  $B_p(b_i)$  is smallest near the RHP poles of  $P(s)$ ). Prop 5A1 shows that the insertion of unstable poles in  $\text{Re}(s) > 0$  into an otherwise stable system always deteriorates the achievable optimal sensitivity.

#### 5B Determination of $\tilde{X}$ and $\tilde{Q}$

The plant  $P(s)$  can be expressed uniquely as the product

$$P(s) = B_z(s) B_p^{-1}(s) P_1(s), \quad (5.9)$$

consisting of the two Blaschke products  $B_z(s)$  and  $B_p(s)$  determined by the  $\text{Re}(s) > 0$  plant zeros and poles respectively, and a factor  $P_1(s)$  which is always in  $H^\infty$ . The burden of the optimization

problem falls on the calculation of the factor of (4.2),

$$\frac{\tilde{X}(s)}{B_p(s)} = D \prod_{i=1}^{r-1} \left( \frac{c_i - s}{\bar{c}_i + s} \right) \quad (5.10)$$

whose coefficients are not known priori.

Assumption (For simplicity) suppose the plant zeros  $b_i$  are distinct.

By Lemma 1,  $X B_p^{-1}$  is unique, and by Prop. 3B1 is determined by the interpolation constraints (3.7), which take the form of  $r$  equations

$$\frac{\tilde{X}(b_j)}{B_p(b_j)} \Delta D \prod_{i=1}^{r-1} \left( \frac{c_i - b_j}{\bar{c}_i + b_j} \right) = \theta_j \quad (5.11)$$

where  $\theta_j \Delta W(b_j)/B_p(b_j)$  are complex constants depending on the values of the weighting  $W(s)$  at the RHP plant zeros, and the location of the plant poles in relation to these zeros.

#### VI EXPLICIT FORMULAS: FEW RHP ZEROS

6A. One RHP Plant-Zero The results here are similar to those of [1], and are omitted from the summary.

#### 6B. Two RHP Plant Zeros

$$\text{In this case, } P(s) = \frac{(b_1 - s)(b_2 - s)}{(b_1 + s)(b_2 + s)} P_1(s) B_p^{-1}(s)$$

where  $b_i$ ,  $i = 1, 2$ , are real or occur in conjugate pairs, and  $\text{Re}(b_i) > 0$ . Here  $r = 2$ . By Lemma 1, the minimal weighted sensitivity function  $\tilde{X}$  must have the form

$$\tilde{X}(s) B_p^{-1}(s) = D(c - s)(c + s)^{-1} \quad (6.1)$$

and the interpolation constraints are

$$D \frac{(c - b_i)}{(c + b_i)} = \theta_i, \quad i = 1, 2. \quad (6.2)$$

If  $D$  is eliminated from the pair of equations (6.2), a quadratic equation is obtained for  $c$ , only one of whose solutions lies in  $\text{Re}(s) \geq 0$ , namely,

$$c = - \frac{(b_2 - b_1)}{2} \left( \frac{\theta_2 + \theta_1}{\theta_2 - \theta_1} \right) + \frac{(b_2 - b_1)^2}{4} \left( \frac{\theta_2 + \theta_1}{\theta_2 - \theta_1} \right)^2 + b_2 b_1 \quad (6.3)$$

If  $D$  is expressed in terms of  $c$  using (6.2) and (6.3), there results

$$D = - \frac{\left( \frac{\theta_2 - \theta_1}{b_2 - b_1} \right) \frac{(b_2 + b_1)}{2}}{\sqrt{\frac{(\theta_2 + \theta_1)}{4} + \left( \frac{\theta_2 - \theta_1}{b_2 - b_1} \right)^2 b_1 b_2}} \quad (6.4)$$

from which it follows that since  $\mu(P) = |p|$ , the minimal sensitivity is

$$\mu(P) = \frac{|\theta_2 - \theta_1|}{|b_2 - b_1|} \frac{|b_2 + b_1|}{2} + \sqrt{\frac{(\theta_1 + \theta_2)^2}{4} + \frac{|\theta_2 - \theta_1|^2}{|b_2 - b_1|^2}} \cdot |b_1 b_2| \quad (6.5)$$

(In the language of [1], (6.5) is the measure of singularity of P.)

It can be deduced from (6.5), after some manipulations, that  $\mu(P) \geq \max(|\theta_1|, |\theta_2|)$  in conformity with Prop. 5A1-a.

If we assume that  $B_p(s) = 1$ , then the optimal improper  $\tilde{Q}$  is

$$\tilde{Q} = [1 - \frac{D(c-s)}{W(s)(c+s)}] \cdot \frac{(b_1+s)(b_2+s)}{(b_1-s)(b_2-s)}$$

If  $W(s)$  is the "lowpass" weighting  $\kappa(s+\kappa)^{-1}$ ,  $\kappa > 0$ , then

$$\tilde{Q}(s) = D \frac{(b_1+s)(b_2+s)}{\kappa(c+s)}$$

which is a lead-type function with an extra "break".

#### VII EXPLICIT FORMULAS: MANY RHP ZEROS

The  $r$  simultaneous equations (5.11) for the coefficients  $c_i$  can be solved by a method based on the Schur-Pick-Nevalinna Theory (See Walsh [2, Ch. 10] for an exposition).

Let  $M$  be fixed, and for any  $|\theta| < M$  let  $U_\theta: H^\infty \rightarrow H^\infty$  denote the mapping  $x \mapsto u$  satisfying the equation

$$u(s) = \frac{M^2[x(s) - \theta]}{M^2 - \bar{\theta}x(s)} \quad (7.1)$$

(7.1) maps the (solid) ball of radius  $|M|$  of  $H^\infty$  into itself, and the set of allpass functions of norm  $|M|$  into itself. Moreover, if  $x(s)$  satisfies an interpolation constraint  $x(b) = \theta$ , then  $u(s)$  has a zero at  $b$ .

(7.1) is now modified by division by  $(b-s)(\bar{b}+s)^{-1}$  thereby removing the zero at  $b$  without increase of norm, to obtain the transformation  $U_{0,b}: H^\infty \rightarrow H^\infty$ ,  $x \mapsto x_1$  which satisfies

$$x_1(s) = \frac{M^2[x(s) - \theta]}{M^2 - \bar{\theta}x(s)} \frac{(\bar{b}+s)}{(b-s)} \quad (7.2)$$

(7.2) still maps the  $|M|$ -ball of  $H^\infty$  into itself. It also establishes a 1:1 correspondence between the set of all allpass functions  $x(s)$  of norm  $|M|$

that assume the value  $\theta$  at  $b$ , and the set of allpass functions  $x_1(s)$  of norm  $|M|$  (whose value at  $b$  is arbitrary). This property of (7.2), together with the result of Lemma 1 that the smallest  $H^\infty$  function satisfying an interpolation constraint is a unique allpass function, i.e., a Balaschke produce, immediately yield the following.

**7.1 prop** If the smallest  $H^\infty$  function assuming the values  $\theta_1, \dots, \theta_r$  at the  $r$  points\*  $(b_1, \dots, b_r)$  is  $X(s)$  and has norm  $|M|$ , then the smallest  $H^\infty$  function assuming the transformed values

$[U_1(\theta_2), U_1(\theta_3), \dots, U_1(\theta_r)], U_1 \triangleq U_{\theta_1, b_1}$ , at the  $(r-1)$  points  $(b_2, \dots, b_r)$  also has norm  $|M|$  and is, uniquely,  $(U_1 X)(s)$ .

Prop 7.1 offers the possibility of reducing any  $H^\infty$  minimization problem involving  $r$  constraints to a problem involving  $(r-1)$  new constraints. If this reduction is repeated  $r$  times, all constraints are removed, and a constant of norm  $|M|$  remains. Let us detail this procedure.

Recall that  $\tilde{x}B_p^{-1}$  is the function of smallest  $H^\infty$  norm assuming the constraint values  $\theta_1, \dots, \theta_r$  at the points  $b_1, \dots, b_r$  in  $\text{Re}(s) > 0$ . Let  $M$  be a complex constant of magnitude  $\|\tilde{x}\|$ , and angle to be determined. Consider an iteration on the constraint set, as follows. Denote the initial constraints by  $\{x_i^{(0)}\}$ , i.e.,

$$\{x_1^{(0)}, \dots, x_r^{(0)}\} = \{\theta_1, \dots, \theta_r\}$$

For any  $i \geq 1$ , if the  $(i-1)$ st constraint set has the form

$$\{x_{i+1}^{(i-1)}, x_{i+1}^{(i-1)}, \dots, x_r^{(i-1)}\}, \text{ let the } i\text{-th constraint}$$

set be

$$\{x_{i+1}^{(i)}, x_{i+2}^{(i)}, \dots, x_r^{(i)}\} = \{T^{(i)}x_{i+1}^{(i-1)}, T^{(i)}x_{i+2}^{(i-1)}, \dots,$$

$$T^{(i)}x_r^{(i-1)}\}$$

where  $T^{(i)}$  denotes the transformation

$$[T^{(i)}x](s) \triangleq \frac{M^2[x(s) - x_i^{(i-1)}]}{M^2 - \bar{x}_i^{(i-1)}x(s)} \frac{(\bar{b}_i + s)}{(b_i - s)}$$

We have the following main result

**Theorem 2** a)  $M$  satisfies the equation

$$[T^{(r)}T^{(r-1)} \dots T^{(1)}](b_r) = M \quad (7.3)$$

(which, after evaluation, yields an algebraic equation in the single variable  $M$ ).

b) The optimal sensitivity function  $\tilde{x}$  is determined explicitly by the equation

$$\tilde{x}(s) = B_p(s) [T^{(1)}]^{-1} [T^{(2)}]^{-1} \dots [T^{(r)}]^{-1} (M)$$

\*Where  $\text{Re } b_i > 0$

where  $[T^{(i)}]^{-1}$  satisfies

$$([T^{(i)}]^{-1}x)(s) = \frac{M^2 \{x(s) + x_1^{(i-1)}\} (b-s)}{[M^2 - \tilde{x}_1^{(i-1)}x(s)] (b+s)}$$

$i = 1, 2, \dots, r$ .

#### VIII COMPARISON WITH WIENER-HOPF QUADRATIC MINIMIZATION

The well known Wiener-Hopf quadratic minimization method, e.g. in Youla et al. [6], provides the optimal feedback for disturbances of fixed power spectrum.

In Fig. 1, let  $d$  be a disturbance obtained by passing unit variance white noise through the filter  $W(s)$ . The mean-square value of the output  $y$  is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \quad (8.1)$$

and the Wiener-Hopf objective is to minimize (8.1) subject to the constraint that  $F$  stabilizes the CL system.

Introduce the Hardy space  $H^2$  of functions  $G(s)$  which are analytic in  $\text{Re}(s) > 0$  and for which the norm is  $\|G\|_2 \triangleq \sup_{\sigma > 0} [\int_{-\infty}^{\infty} |G(\sigma + j\omega)|^2 d\omega]^{1/2}$ . This

norm can be shown to be computable on the  $j\omega$ -axis, and  $\|G\|_2 = [\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega]^{1/2}$ .

The Wiener-Hopf problem is equivalent to the problem of minimizing the  $H^2$  norm of the weighted sensitivity function  $X$ , with the same constraints as in Prop 3B1, and can be solved by the same method as the  $H^\infty$  problem. The result is that there is a unique weighted sensitivity function  $\tilde{X}$  in  $H^2$  of minimum norm, which is characterized by the formula\*

$$\tilde{X}(s) = \frac{\phi(s)}{\prod_{i=1}^r (b_i + s)} \prod_{j=1}^q \frac{a_j - s}{a_j + s} \quad (8.2)$$

where  $\phi(s)$  is a polynomial of degree  $< r$ , which is uniquely determined by the plant-zero interpolation constraints (3.7).  $\phi(s)$  can be computed, e.g., by the Lagrange interpolation formula. (Eq. (8.2) is equivalent to (41) of [6].)

#### 8A. An Example

Let us consider a representative servo problem involving a single RHP plant zero, and compare the  $H^\infty$  and  $H^2$  solutions.

As RHP plant poles affect both solutions similarly (cf. (8.2) and (4.2)), the plant will be assumed stable. Suppose that

$$P(s) = (b-s)(b+s)^{-1} P_1(s), \quad b > 0, P_1 \in H^\infty$$

Typically, servomechanisms must be able to re-

spond to disturbance having not one but a variety of power spectra. Low frequency disturbances, which must be attenuated, occur most often. However, even very high frequency disturbances appear occasionally, and response to them must be limited, even though attenuation may be impractical. This situation is tailor-made for an  $H^\infty$  description.

Suppose that disturbance spectra can be arbitrary subject to the upper bound  $|d(j\omega)| \leq |W(j\omega)|$ , where

$$W(s) = \frac{0.1b + \xi s}{0.1b + s}^3, \quad \xi = \left(\frac{1}{20}\right)^{1/3}$$

The weighting here is substantial for frequencies smaller than  $b$ , ( $|W(j\omega)| > 0.707$  for  $|\omega| \in [0, 0.1b]$ ) then drops off quite rapidly, but never falls below the lower bound  $\xi^3 = 1/20$ . The optimal unweighted sensitivity function for this problem, obtained as in Sect. 6A, is

$$(1 - P\tilde{Q}) = W(b)/W(s) = 0.077 \left(\frac{0.1b + s}{0.1b + \xi s}\right)^3$$

and is illustrated in Fig. 3.

Although there is no completely satisfactory description of this problem in  $H^2$ , a widely used current approach is to design  $X$  as though  $|W(s)|$  were a fixed power spectrum. Strictly speaking this is possible only if  $W(j\omega) \in H^2$ , i.e., if  $\xi = 0$ . In that case  $r = 1$  in (8.2),  $\xi(s)$  is a constant, and on matching interpolation constraints, the result

$$(1 - P\tilde{Q}) = \left[W(b)/W(s)\right] \frac{2b}{s+b} = (1 - P\tilde{Q}) \frac{2b}{s+b} \quad (8.3)$$

is obtained. The unweighted sensitivity function is greater by a factor of nearly two over all frequencies of importance than in the  $H^\infty$  case. If  $\xi \neq 0$ ,  $W(s)$  can be approximated in a distributional sense by a sequence of  $H^2$  functions, and the result (8.3) and conclusion still hold in the limit, as shown in Fig. 3.

**Remark.** The  $H^2$  method forces the integral-squared value of the unweighted sensitivity function to be small over high frequencies (i.e., over  $(b, \infty)$ ) where in fact the integral is of no consequence (indeed, in practice can not be finite, as  $P\tilde{Q}$  is strictly proper), and where in reality it is enough to maintain an upper-bound on that unweighted sensitivity function. This gratuitous reduction comes at the expense of the sensitivity at all frequencies of importance.

#### 8B. Conclusions

If the disturbance power-spectrum exactly equals  $|W(j\omega)|$ , quadratic optimization gives the best sensitivity function. However, this sensitivity function can be quite poor for other spectra, even within the passband of  $W(j\omega)$ . For example, it is poor for a disturbance whose power is concentrated in a narrow band around the frequency of the peak in Fig. 3.

\*For plants with conjugate symmetry.

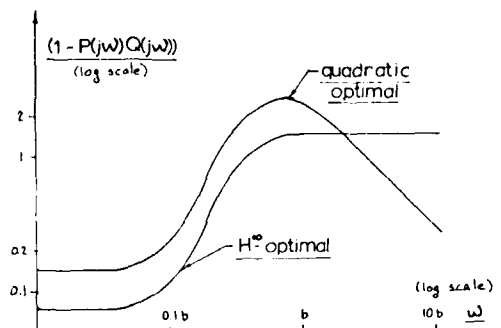


Fig.3 Unweighted optimal sensitivity functions

$H^\infty$  optimization, on the other hand, gives a sensitivity function optimized for a set of disturbance power-spectra, namely, those bounded by  $|W(j\omega)|$  (but arbitrary otherwise).

Of course, neither the  $H^\infty$  nor the quadratic solutions presented here consider practical issues of bandwidth cost, power cost, etc. However, the  $H^\infty$  formulation appears better able to cope with those systems that are required to tolerate large classes of disturbance power-spectra, as feedback servo-mechanisms are (and as opposed, say, to single-source communications systems).

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function matrices  $G(\lambda)$  of such plants. The usefulness of the resulting simple closed-form formula for  $K(\lambda)$  is illustrated by designing an error-actuated controller for an open-loop unstable chemical reactor which results in superior closed-loop behavior to that obtained by using controllers for the same plant designed by alternative techniques.

79/00/00 81A20455

UTTL: A complex variable approach to the analysis of linear multivariable feedback systems  
A/POSTLETHWAITE, I.; B/MACFARLANE, A. G. J. PAA:  
B/(Cambridge University, Cambridge, England) Berlin, Springer-Verlag (Lecture Notes in Control and Information Sciences, Volume 12), 1979, 181 p.

ABS: The aim of the present work is to extend the concepts underlying the techniques of Nyquist, Bode, and Evans to multivariable systems. The study of complex gain as a function of complex frequency and complex frequency as a function of complex gain are extended to the multivariable case by associating with transfer function matrices a pair of analytic functions: a characteristic gain function and a characteristic frequency function. Further, a comprehensive discussion of the generalized Nyquist stability criterion for multivariable feedback systems is presented along with a generalization of the inverse Nyquist stability criterion to the multivariable case. Finally, the Evans' root locus approach is extended to multivariable systems. 79/00/00 80A40827

UTTL: The deadbeat control of linear multivariable systems with inaccessible state  
A/OREILLY, J. PAA: A/(Liverpool, University, Liverpool, England) International Journal of Control, vol. 31, Apr. 1980, p. 645-654.

ABS: The problem of deadbeat controller design for a class of discrete-time linear multivariable systems with inaccessible state is considered. A deadbeat controller, based upon minimal-order observer, and specified by two gain matrices in compact form, is presented which can drive the system state from any arbitrary initial state to the origin in at most  $v(c) + v(o) - 1$  steps where  $v(c)$  and  $v(o)$  respectively are the controllability and observability indices of the system. The simplicity and generality of the design is demonstrated by numerical example. 80/04/00 80A36644

UTTL: Some unifying concepts in multivariable feedback design

AUTH: A/OWENS, D. H. PAA: A/(Sheffield, University, Sheffield, England) International Journal of Control, vol. 33, Apr. 1981, p. 701-711.

ABS: Frequency domain techniques for computer-aided design of multivariable feedback systems are now well established in the form of several, apparently distinct, design techniques. It is shown that a unified design structure can be developed based on the theoretical concepts of precompensation, eigenvalue approximation and permissible, constant, input/output transformations. 81/04/00 81A28990

UTTL: Parameter adaptive control of linear multivariable systems

AUTH: A/ELLIOTT, H.; B/WOLOVICH, W. A. PAA: A/(Colorado State University, Fort Collins, Colo.); B/(Brown University, Providence, R.I.) In: Asilomar Conference on Circuits, Systems, and Computers, 13th, Pacific Grove, Calif., November 5-7, 1979, Conference Record, (AB1-26601 10-63) Piscataway, N.J., Institute of Electrical and Electronics Engineers, Inc., 1980, p. 33-37.

ABS: This report presents an adaptive control structure which can be used to assign all poles and zeros of a continuous time linear multivariable system represented by a (pxm), strictly proper transfer matrix  $T(s)$ , provided  $p$  is not less than  $m$  and  $T(s)$  has no right half plane zeros. The controller parameters can be directly estimated from input-output data. The report also serves to point out the type of a priori information necessary for multivariable adaptive controller design. This information is a natural extension of that required in the scalar case. 80/00/00 81A26608

UTTL: High-gain error-actuated controllers for a class of linear multivariable plants

AUTH: A/PORTER, B.; B/BRADSHAW, A. PAA: B/(Salford, University, Salford, Lancs., England) In: Conference on Decision and Control, and Symposium on Adaptive Processes, 18th, Fort Lauderdale, Fla., December 12-14, 1979, Proceedings, Volume 1, (AB1-20426 07-63) Piscataway, N.J., Institute of Electrical and Electronics Engineers, Inc., 1979, p. 322-325.

ABS: The transfer function matrices  $K(\lambda)$  of high-performance, high-gain error-actuated controllers for a practically important class of controllable and observable linear multivariable plants are expressed directly in terms of properties of the transfer



UTTL: Nested-feedback-loop decomposition for reduction of linear multivariable systems

AUTH: A/OWENS, D. H. PAA: A/Sheffield, University, Sheffield, England) IEE Proceedings, Part D - Control Theory and Applications, vol. 127, pt. D, no. 3, May 1980, p. 103-114.

ABS: Previous results on the reduction of single-input/single-output systems are extended to the case of square invertible multivariable systems. It is shown that such systems have a unique decomposition in the form of a forward and a feedback path each having special properties relating to the invariant zeros of the original system. Under certain generic conditions, this decomposition can be extended to yield a representation of the system as a nested sequence of feedback loops. This provides a convenient method of deriving reduced-order models, which will give good matching of the asymptotic system root locus and open-loop and closed-loop system dynamics, and hence will be a valid tool in closed-loop controller design. 80/05/00 80A36304

UTTL: Necessary conditions for absolute stability of nonlinear multivariable regulators

AUTH: A/PORTER, B. PAA: A/Salford, University, Salford, England) Electronics Letters, vol. 16, May 9, 1980, p. 392, 393.

ABS: Simple necessary conditions involving the fundamental frequency-domain concept of transmission zeros are established for the absolute stability of nonlinear multivariable regulators. These results significantly complement the sufficient conditions provided by the matrix Popov criterion in the analysis and synthesis of nonlinear multivariable control systems. 80/05/09 80A36C24

UTTL: Linear multivariable control - A geometric approach /2nd edition/

AUTH: A/WORHAM, W. M. PAA: A/Toronto, University, Toronto, Canada) New York, Springer-Verlag New York, Inc. (Applications of Mathematics, Volume 10), 1979, 340 p.

ABS: The work presents a geometric approach to the structural synthesis of multivariable control systems that are linear, time-invariant and of finite dynamic order. It is addressed mainly to students specializing in control systems research and development, and to mathematicians with some previous acquaintance with control problems. Subjects include disturbance decoupling and output stabilization as well as tracking and regulation. Other topics are discussed such as noninteracting control and quadratic

optimization. 79/00/00 80A28955

UTTL: Multivariable synthesis with inverses A/PECZKOWSKI, J. L. B/SAIN, M. K. C/LEAKE, R. J. PAA: A/Bendix Corp., South Bend, Ind.) B/Notre Dame, University, Notre Dame, Ind.) C/California State University, Fresno, Calif.) In: Joint Automatic Control Conference, Denver, Colo., June 17-21, 1979, Proceedings. (ABO-24226 08-63) New York, American Institute of Chemical Engineers, 1979, p. 375-380.

ABS: The application of total synthesis (TS) methods to the design of controller dynamics for linear multivariable models of realistic turbine engine simulations is illustrated. TS methods provide the designer with a capability to specify thoroughly and directly the nominal dynamic relationship between command or request variables and controlled or response variables. Under reasonable assumptions, this capability can include transient response as well as limiting values, and of course inter al stability. Particular stress is placed upon the inverse total synthesis problem (ITSP), which emphasizes the inverse of the plant input/output relation, expressed typically as a matrix of transfer functions. In numerous case studies, the ITS approach has shown an ability to preserve designer insight and influence, and has turned out to be relatively easy to understand; both properties are of importance for general control applications. 79/00/00 80A24246

UTTL: On hidden stability margins in multivariable control

AUTH: A/REKASIUS, Z. V. PAA: A/Northwestern University, Evanston, Ill.) In: Joint Automatic Control Conference, Denver, Colo., June 17-21, 1979, Proceedings. (ABO-24226 08-63) New York, American Institute of Chemical Engineers, 1979, p. 354-357.

ABS: A theoretically very appealing design procedure in multivariable control is first to decouple the multivariable system into single-input single-output subsystems and then to design controllers for each subsystem separately. However, decoupled systems may have very low stability margins and become unstable under slight changes in plant or controller parameters, even if all the single-input single-output subsystems exhibit individually very high stability margins. It is shown in the paper how stability margins and peak resonance can be computed at several points in the system such that the hidden stability exhibit individually very high stability margins. 79/00/00 80A24243

UTTL: Optimization of multivariable dynamic systems  
 AUTH: A/MEEROV, M. V. (Automatika i Telemekhanika, July 1979, p. 36-42.) Automation and Remote Control, vol. 40, no. 7, Dec. 10, 1979, pt. 1, p. 975-980. Translation.

ABS: Optimization of linear multivariable dynamic systems that have a specific structure is considered. By studying this specific structure, it is possible to obtain an optimization method that is a continuous counterpart of the simplex method of linear programming. 79/12/10 80A23048

UTTL: Design methods of optimal feedback control for multivariable control systems  
 AUTH: A/NAEIJDE, W. J. Delft, Technische Hogeschool, Doctor in de technische Wetenschappen Dissertation, 1979, 166 p. In Dutch.

ABS: The optimization of cost functions is found to be expedient in the design process of multivariable industrial systems. A filter is constructed that generates a set of time functions on the basis of exact measurements of all necessary state variables. These measurements are determined such that the time functions are as close as possible to the true values of the physical plant state. It is determined that minimization of the error variances leads to optimal state feedback laws in conjunction with a state reconstruction filter. It is shown that this design method lacks the flexibility needed to handle certain kinds of engineering constraints, including the fact that the controller must not be too complicated, must have high integrity and must not introduce too much phase-advance in any loop. 79/00/00 80A15652

UTTL: Solution methods for one class of optimal control in distributed multivariable systems in the presence of phase constraints

AUTH: A/MEEROV, M. V.; B/LITVAK, M. L. PIA: B/(Academy of Sciences, Institute of Control Sciences, Moscow, USSR) In: A link between science and applications of automatic control; Proceedings of the Seventh Triennial World Congress, Helsinki, Finland, June 12-16, 1978, Volume 3. (A80-14794 03-63) Oxford and New York, Pergamon Press, 1979, p. 2103-2106. 79/00/00 80A14872

UTTL: Stabilization by digital controllers of multivariable linear systems with time-lags  
 AUTH: A/BALLESTRINO, A.; B/CELENTANO, G. PAA: B/(Napoli, Università, Naples, Italy) In: A link between science and applications of automatic control, Proceedings of the Seventh Triennial World Congress, Helsinki, Finland, June 12-16, 1978, Volume 3. (A80-14794 03-63) Oxford and New York, Pergamon Press, 1979, p. 2091-2094.

ABS: The problem of stabilizing linear systems with time-lags is investigated. Under rather general assumptions it is shown that, by means of a digital controller, the system can be always reduced to a system with mere delay connected in cascade with a subsystem whose poles can be arbitrarily assigned. 79/00/00 80A14870

UTTL: Design of multivariable discrete-time regulators  
 AUTH: A/JAAKSOO, U. PAA: A/(Akademia Nauk Estoniskoi SSR, Institut Kibernetiki, Tallin, Estonian SSR) In: A link between science and applications of automatic control; Proceedings of the Seventh Triennial World Congress, Helsinki, Finland, June 12-16, 1978, Volume 3. (A80-14794 03-63) Oxford and New York, Pergamon Press, 1979, p. 1831-1836.

ABS: A method for designing multivariable discrete time regulators is presented and two different types of regulators are discussed: optimal state proportional plus integral and output proportional plus integral. A new method for determining the output regulator is also presented. This method is based on a singular value decomposition of the optimal closed loop system matrix. It is determined that this method leads to a weighted least squares solution of a linear matrix equation, and guarantees a good approximation of the optimal closed loop system. 79/00/00 80A14853

UTTL: Structural reliability and robustness properties of optimal linear-quadratic multivariable regulators  
 AUTH: A/WONG, P.-K.; B/STEIN, G.; C/ATHANS, M. PAA: C/(MIT, Cambridge, Mass.) CORP. Massachusetts Inst. of Tech., Cambridge. In: A link between science and applications of automatic control; Proceedings of the Seventh Triennial World Congress, Helsinki, Finland, June 12-16, 1978, Volume 3. (A80-14794 03-63) Oxford and New York, Pergamon Press, 1979, p. 1797-1805.

ABS: Strong sufficient conditions are derived for the robustness of optimal linear-quadratic (LQ) regulators to large parameter perturbations. In particular, it is shown that under certain conditions LQ designs remain stable in the presence of actuator channel failures. The general results can be specialized to provide

insight into the gain margin, gain reduction, and phase margin properties of optimal LO regulators. 79/00/00 80A14850

UTTL: The relative stability margins of multivariable systems - A characteristic locus approach  
AUTH: A/CAVERON, R.; B/KOUVARITAKIS, B. PAA: B/(Bradford, University, Bradford, Yorks., England) International Journal of Control, vol. 30, Oct. 1979, p. 629-651.

ABS: Recent work has proven that characteristic locus plots form the natural medium for the generalization of the Nyquist approach. In the present paper these plots are used to extend classical scalar techniques of assessing relative stability margins to the multivariable case. Thus the estimation of closed-loop poles using curvilinear squares is first discussed and subsequently the use of constant dynamic magnification circles in predicting performance under feedback is considered. A new concept relevant to both techniques is introduced, namely that of interference. Interference relates to the loop distribution of eigenvalues, and complements the concept of interaction which relates to eigenvector distribution. 79/10/00 80A12991

UTTL: Robust stability of multivariable feedback systems  
AUTH: A/SANDELL, N. R., JR. PAA: A/(MIT, Cambridge, Mass.) In: Annual Allerton Conference on Communication, Control and Computing, 16th, Monticello, Ill., October 4-6, 1978, Proceedings. (A80-12690 02-63) Urbana, Ill.: University of Illinois, 1978, p. 471-479.

ABS: The paper presents a simple approach to determining conditions for stability of multivariable feedback systems subject to additive and multiplicative perturbations in the operators describing these systems. The approach is based on standard techniques used in input-output stability theory, and provides an alternative development and generalization of some conditions for the time-invariant case that have appeared in the literature very recently. 79/00/00 80A12717

UTTL: New dominance characteristics for the multivariable Nyquist array method  
AUTH: A/LEININGER, G. G. PAA: A/(Purdue University, West Lafayette, Ind.) CORP: Purdue Univ., Lafayette, Ind. International Journal of Control, vol. 30, Sept. 1979, p. 459-475.

ABS: Three new dominance characteristics are introduced to

the multivariable Nyquist array methods. The first characteristic utilizes a conformal mapping procedure to establish a new set of bands in the 'image' plane to assist in feedback gain selection and stability assessment. The second characteristic uses the 'image' band concept to provide a theoretical foundation for finite frequency dominance considerations, thus removing the restrictive requirement of dominance for all  $s$  on the Nyquist D contour. The third characteristic provides for the sharing of system dominance among the feedback control loops. The dominance sharing concept may be used to establish dominance and/or to improve the dominance condition in prespecified feedback loops. 79/09/00 79A52066

UTTL: Bounded input-bounded output stability of multivariable control systems with many nonlinearities  
AUTH: A/JOSSELYN, R.; B/RAJU, G. V. S. PAA: A/(Honeywell, Inc., Defense Systems Div., Minneapolis, Minn.); B/(Ohio University, Athens, Ohio) In: Midwest Symposium on Circuits and Systems, 21st, Ames, Iowa, August 14, 15, 1978, Proceedings. (A79-50051 22-63) North Hollywood, Calif.: Western Periodicals Co., 1978, p. 495-499.

ABS: Sufficient conditions are presented for the boundary-input boundary-output stability of a multivariable control system with many nonlinearities that are sector restricted. Assumptions are not made, however, regarding controllability and observability. Yet it is determined that if a certain stability matrix is positive semidefinite, then BIBO stability results. A similar result is obtained for monotonic nonlinearities with the additional conclusion that the bounded limit state is always unique and periodic if the input is periodic. 78/00/00 79A50068

UTTL: Boundedness properties of nonlinear multivariable feedback systems  
AUTH: A/ATHERTON, D. P.; B/OWENS, D. H. PAA: B/(Sheffield, University, Sheffield, England) Electronics Letters, vol. 15, Aug. 30, 1979, p. 559-561.

ABS: A boundedness theorem is derived for a commonly encountered class of memoryless nonlinearities in multivariable feedback systems. The result can be useful when absolute stability cannot be proved or when investigating the existence of limit cycles. 79/08/30 79A49463

**UTTL:** Output regulation and tracking in linear multivariable systems  
**AUTH:** A/WOLOVICH, W. A.; B/PEREIRA, P. PAA: A/(Brown University, Providence, R.I.) In: 1978 Conference on Decision and Control, 17th, San Diego, Calif., January 10-12, 1979. Proceedings (A79-47930 21-63) New York, Institute of Electrical and Electronics Engineers, Inc., 1979, p. 801-808. Conselho Nacional de Pesquisas

**ABS:** It is shown that the related questions of output tracking and output regulation (with and without input sensor measurement), all with associated internal stability, can be resolved via a single polynomial matrix formulation which employs the recently developed notion of skew prime polynomial matrices. In particular, the primary condition which must be satisfied in all cases is that  $R(s)$ , the plant numerator matrix, and a matrix whose zeros represent the modes of the exogenous system, be externally skew prime. This observation is then elucidated in light of a new internal model principle. 79/00/CO 79A47979

**UTTL:** Triangularization technique for the design of

multivariable control systems  
**AUTH:** A/HUGG, N. T.; B/ANDERSON, B. D. O. PAA: B/(Newcastle University, Newcastle, New South Wales, Australia) In: 1978 Conference on Decision and Control, 17th, San Diego, Calif., January 10-12, 1979. Proceedings (A79-47930 21-63) New York, Institute of Electrical and Electronics Engineers, Inc., 1979, p. 795-800.

**ABS:** The paper presents a novel technique for the design of multivariable control systems. A stable and proper precompensator is to be determined for a multivariable plant such that the compensated plant transfer function matrix is triangular and diagonally dominant in a nonstandard way. As a consequence of the triangular-diagonal-dominance property, only the diagonal elements need to be considered in an over-all closed-loop design. In effect, the technique provides a systematic procedure to reduce a multivariable design problem to independent scalar design problems. 79/00/CO 79A47978

**UTTL:** Stability criteria for multivariable non-linear feedback systems consisting of low order subsystems  
**AUTH:** A/SAEKI, M.; B/ARAKI, M.; C/KONDO, B. PAA: C/(Kyoto University, Kyoto, Japan) International Journal of Control, vol. 29, May 1979, p. 733-747.

**ABS:** Popov-type stability criteria are given for two types of multivariable feedback systems. The systems are regarded as interconnections of several subsystems and

the composite-system method is used to derive the criteria. The values of constants which are needed in the criteria are calculated and tabulated for first- and second-order subsystems. By using these values, we can immediately obtain a stability criterion for the system consisting of first- and/or second-order subsystems. Examples show that the criteria obtained are as sharp as or sharper than previously reported criteria. 79/05/00 79A41272

**UTTL:** Structural properties of multivariable linear systems

**AUTH:** A/DENHAM, M. J.; B/VARDOLAKIS, A. I. G. PAA: A/(Kingston Polytechnic, Kingston-on-Thames, Surrey, England); B/(Cambridge University, Cambridge, England); Institution of Electrical Engineers, Proceedings, vol. 126, June 1979, p. 591-595.

**ABS:** The design of controllers for multivariable systems, in particular large-scale systems, requires knowledge and exploitation of the structural properties of the systems. The ability to achieve certain structural properties, e.g. regulation, tracking, noninteraction, disturbance, rejection etc., can be determined on the basis of a small number of basic system concepts. In this paper, structural decomposition of a multivariable system is studied using the concept of a controllability subspace, and a review of recent work to investigate this concept is given. 79/06/00 79A41182

**UTTL:** Sequential design of linear multivariable systems

**AUTH:** A/MAYNE, D. O. PAA: A/(Imperial College of Science and Technology, London, England) Institution of Electrical Engineers, Proceedings, vol. 126, June 1979, p. 568-572.

**ABS:** The paper describes and critically assesses, in the light of recent advances, the sequential return difference method for the computer-aided design of linear multivariable control systems. In this method, a sequence of single-loop designs, using classical procedures, yields a multivariable design satisfying various criteria such as stability, disturbance attenuation, low interaction and integrity. 79/06/00 79A41180

**UTTL:** Output regulation and tracking in linear multivariable systems

**AUTH:** A/WOLOVICH, W. A.; B/FERREIRA, P. PAA: A/(Brown University, Providence, R.I.) IEEE Transactions on Automatic Control, vol. AC-24, June 1979, p. 460-465. Conselho Nacional de Pesquisas

**ABS:** It is shown that the related questions of output tracking and output regulation (with and without input sensor measurement), all with associated internal stability, can be resolved via a single polynomial matrix formulation which employs the recently developed notion of skew prime polynomial matrices. In particular, the primary condition which must be satisfied in all cases is that  $R(s)$ , the plant numerator matrix, and a matrix whose zeros represent the modes of the exogenous system, be externally skew prime. This observation is then elucidated in light of a new internal model principle. 79/06/00 79A40097

**UTTL:** A pole assignment algorithm for multivariable control systems

**AUTH:** A/LEE, G.; B/JORDAN, D.; C/SHRWARDY, M. PAA: A/(Colorado State University, Fort Collins, Colo.); B/(Connecticut University, Storrs, Conn.); C/(Bochum, Ruhr-Universitaet, Bochum, West Germany) IEEE Transactions on Automatic Control, vol. AC-24, Apr. 1979, p. 357-362.

**ABS:** A general compensation design algorithm for pole assignment is considered. The methodology presented is an extension of previous work in which plant and compensator dynamics are described in such a way that the influence of compensation on the poles of the system is directly observed. By holding some of the poles of the plant and compensator stationary at each iteration, the design problem is transformed into an equivalent multi-input, single-output system to which root locus techniques can be directly applied. The design algorithm is iterative in nature and allows the designer the freedom to use his expertise within a standardized framework. An example of a general compensation problem is given to illustrate the simplicity of the design method. 79/04/00 79A34261

**UTTL:** Decentralized transmission zeros of linear multivariable continuous-time systems

**AUTH:** A/PORTER, B.; B/BRADSHAW, A. PAA: B/(Salford, University, Salford, Lancs., England) Electronics Letters, vol. 15, Apr. 12, 1979, p. 230, 231. 79/04/12 79A33810

**UTTL:** The design of multivariable systems having zero sensitive poles

**AUTH:** A/SHAKED, U. PAA: A/(Tel Aviv University, Tel Aviv, Israel) IEEE Transactions on Automatic Control, vol. AC-24, Feb. 1979, p. 117-119.

**ABS:** A design technique that assigns the maximal number of poles of a linear multivariable system and makes them insensitive to uncertainty in the parameters of the system dynamical matrix is introduced. Its immediate application is in multivariable dominant poles design. 79/02/00 79A24548

**UTTL:** Approximate decoupling by dynamic compensation

**AUTH:** A/COOK, P. A. PAA: A/(University of Manchester Institute of Science and Technology, Manchester, England) International Journal of Control, vol. 28, Dec. 1978, p. 847-852.

**ABS:** This paper proposes a method for designing a dynamic precompensator for a multivariable linear system, so as to make the transfer-function matrix of the compensated system diagonal at a specified set of frequencies. The properties of the resulting compensator are investigated, together with the approach to exact diagonalization as the complexity of the structure is increased. 78/12/00 79A23822

**UTTL:** A sensitivity analysis for the F100 turbofan engine using the multivariable Nyquist array

**AUTH:** A/LEININGER, G. G.; B/BORSYIAK, M. L. PAA: B/(Toledo, University, Toledo, Ohio) CORP: Toledo Univ., Ohio. In: Modeling and simulation, Volume 9 - Proceedings of the Ninth Annual Pittsburgh Conference, Pittsburgh, Pa., April 27, 28, 1978, Part 4. (A79-23776 (8-59) Pittsburgh, Pa.: Instrument Society of America, 1978, p. 1327-1333.

**ABS:** In the feedback control design of multivariable systems, closed loop performance evaluations must include the dynamic behavior of variables unavailable to the feedback controller. For the multivariable Nyquist array method, a set of sensitivity functions are proposed to simplify the adjustment of compensator parameters when the dynamic response of the unmeasurable output variables is unacceptable. A sensitivity study to improve thrust and turbine temperature responses for the Pratt-Whitney F100 turbofan engine demonstrates the utility of the proposed method. 78/00/00 79A23797

UTTL: Design of multivariable variable structure model following control systems

AUTH: A/YOUNG, K.-K. D. PAA: A/(Illinois, University, Urbana, Ill.) In: Conference on Decision and Control, and Symposium on Adaptive Processes, 16th, and Special Symposium on Fuzzy Set Theory and Applications, New Orleans, La., December 7-9, 1977, Proceedings, Volume 1. (A79-14957 04-63) Piscataway, N.J.: Institute of Electrical and Electronics Engineers, Inc., 1977, p. 1042-1048.

ABS: A design procedure for model following control systems is developed by utilizing sliding mode in variable structure systems. It is shown that a variable structure model following control system (VSMFC) possesses an adaption mechanism similar to that in adaptive model following control systems. For some classes of plant parameter variations, shaping of the error transients can be achieved systematically using this design. Two control problems which have been approached via adaptive model following technique are considered and performance comparison with the present design is made. 77/00/00 79A15010

UTTL: Perfect regulation and feedforward control of linear multivariable systems

AUTH: A/FRANCIS, B. A. PAA: A/(McGill University, Montreal, Canada) In: Conference on Decision and Control, and Symposium on Adaptive Processes, 16th, and Special Symposium on Fuzzy Set Theory and Applications, New Orleans, La., December 7-9, 1977, Proceedings, Volume 1. (A79-14957 04-63) Piscataway, N.J.: Institute of Electrical and Electronics Engineers, Inc., 1977, p. 760-762.

ABS: The time-invariant linear-quadratic regulator is considered as the weighting on the control energy tends to zero (cheap control). A geometric condition is found which characterizes when the cost tends to zero (perfect regulation) and this is shown to be equivalent to a special feedforward control problem. The results are extended to the steady-state LQG problem with cheap control and accurate observations 77/00/00 79A15002

UTTL: Determination of the structure of multivariable stochastic linear systems

AUTH: A/SUEN, L. C.; B/LIU, R. PAA: A/(Bell Telephone Laboratories, Inc., Naperville, Ill.) In: Conference on Decision and Control, and Symposium on Adaptive Processes, 16th, and Special Symposium on Fuzzy Set Theory and Applications, New Orleans, La., December 7-9, 1977, Proceedings, Volume 1. (A79-14957 04-63) Piscataway, N.J.: Institute of Electrical and

Electronics Engineers, Inc., 1977, p. 429-434.  
The problem of determination of the structure of a multivariable linear system from a given input-output data is considered, for both deterministic and stochastic cases. A new canonical input-output form is proposed for this study. A systematic procedure, normalized residual technique, for the estimation of the structure of the canonical form is presented. A comparison of our result with others is discussed. 77/00/00 79A14981

UTTL: On alternative methodologies for the design of robust linear multivariable regulators  
AUTH: A/KWATNY, H. G.; B/KALNITSKY, K. C. PAA: A/(Drexel University, Philadelphia, Pa.); B/(Analytic Sciences Corp., Reading, Mass.) IEEE Transactions on Automatic Control, vol. AC-23, Oct. 1978, p. 930-933.  
This paper presents two synthesis algorithms which embody the two major variants of the numerous methodologies which have been proposed for the design of multivariable linear regulators which exhibit the property of disturbance rejection with or without additional robustness qualities. It is shown that these two procedures generally lead to substantively different compensator structures. 78/10/00 79A12289

UTTL: Some sufficient and some necessary conditions for the stability of multivariable systems  
AUTH: A/SHIEH, L. S.; B/YATES, R. E.; C/SHIH, C. D. PAA: A/(Houston, University, Houston, Tex.); B/(U.S. Army, Guidance and Control Directorate, Redstone Arsenal, Ala.) ASME, Transactions, Journal of Dynamic Systems, Measurement, and Control, vol. 100, Sept. 1978, p. 214-218.

ABS: Some sufficient and some necessary conditions for the stability of a class of multivariable systems represented by matrix polynomials are derived. A new linear block transformation is also established for transforming an observable block companion form to the block Schwarz form. 78/09/00 79A12125

UTTL: The use of frequency transmission concepts in linear multivariable system analysis

AUTH: A/MARCANIAS, N.; B/KOUVARITAKIS, B. PAA: A/(Cambridge University, Cambridge, England); B/(Bradford, University, Bradford, England) International Journal of Control, vol. 28, Aug. 1978, p. 197-240.

ABS: The study of the transmission of a particular frequency or of a set of particular frequencies leads

to the definition of monofrequency transmission subspaces or multifrequency transmission subspaces, respectively. Further investigation of the properties of these subspaces yields insight into the geometric structure of linear multivariable systems and suggests practical techniques for the feedback designer. One such technique concerns the placement of eigenvectors and provides the geometric means for the derivation of solutions to the pole-placement problem. 78/08/00 78A50270

UTTL: The optimal root loci of linear multivariable systems

AUTH: A/KOUVARITAKIS, B. PAA: A/(Bradford, University, Bradford, England) International Journal of Control, vol. 28, July 1978, p. 33-62.

ABS: The movement of the closed-loop poles of optimal systems when the weight on the control effort in the index of performance is relaxed to arbitrarily low levels is studied. A new root-locus approach is used to study all orders of behavior. The results derived relate the asymptotic behavior of the poles of the optimal system to that of the original system, and for this reason more light is cast onto the function of the optimal controller. Explicit results are given for all orders up to five although the symmetric nature of the pattern of behavior enables by conjecture the extension of these results to all orders. A numerical example is included. 78/07/00 78A47478

UTTL: The asymptotic behavior of the root-loci of multivariable optimal regulators

AUTH: A/SHAKED, U. PAA: A/(Tel Aviv, University, Tel Aviv, Israel) IEEE Transactions on Automatic Control, vol. AC-23, June 1978, p. 425-430.

ABS: The loci of the closed-loop poles of the multivariable, time-invariant, linear optimal regulator are shown to group into the left half-plane part of several Butterworth configurations as the weight on the input in the criterion approaches zero. It is proved that these configurations are of even order and that they are always centered at the origin. The number of configurations of any even order, their radii, and the angle of their corresponding asymptotes are expressed in terms of the criterion and the system constant matrices. 78/06/00 78A41979

UTTL: Singular perturbation methods in the design of linear multivariable tracking systems for plants with polynomial command inputs

AUTH: A/PORTER, B. B/TSINGAS, A. PAA: B/(Salford, University, Salford, Lancs., England, International Journal of Control, vol. 27, Apr. 1978, p. 639-650. 78/04/00 78A36286

UTTL: A note on diagonal dominance and the stability of multivariable feedback systems

AUTH: A/OWENS, D. H. PAA: A/(Sheffield, University, Sheffield, England) International Journal of Control, vol. 27, Apr. 1978, p. 603-608.

ABS: Diagonal dominance plays a fundamental role in the design of multivariable feedback control systems by the method of dyadic expansion and the inverse Nyquist array by providing a systematic procedure for the structural simplification of the return-difference determinant. It is shown that, by the use of equivalence transformations and origin shift methods, sufficient conditions for closed-loop stability using diagonal dominance methods can be obtained which remove many of the difficulties arising in previous formulations. 78/04/00 78A36287

UTTL: Steady-state decoupling of linear multivariable systems

AUTH: A/HUANG, J. Y. B/THALER, G. J. PAA: A/(Santa Clara, University, Santa Clara, Calif.), B/(U.S. Naval Postgraduate School, Monterey, Calif.) International Journal of Control, vol. 27, Apr. 1978, p. 579-587.

ABS: A constructive criterion for decoupling the steady states of a linear, time-invariant multivariable system is developed. This criterion consists of a set of inequalities which, when satisfied, will cause the steady states to be decoupled. It turns out that pure integrators in the loops play an important role. 78/04/00 78A36286

UTTL: Sensitivity reduction in exact model-matching of linear multivariable systems

AUTH: A/PARASKEVOPOULOS, P. N. PAA: A/(Thrace, University, Xanthi; Democritus Nuclear Research Centre, Athens, Greece) Franklin Institute, Journal, vol. 305, Feb. 1978, p. 99-118.

ABS: This paper is concerned with the problem of reducing the effect of parameter variations (disturbances) of the open-loop system on the exact model-matching of linear multivariable systems. The proposed method determines a controller such that the input-output

description of the closed-loop system matches that of the model to second or higher order in the variations of the parameters of the open-loop system. These results appear to be the first in the field of exact model-matching. Since in most realistic cases the parameters of the open-loop system undergo disturbances, the present results could be very useful in many practical exact model-matching design problems. Several examples are included to illustrate the proposed method. 78/02/00 78A32323

**UTTL:** The structure of multivariable systems under the action of constant output feedback control laws  
**AUTH:** A/VARDULAKIS, A. I. PAA: A/Imperial College of Science and Technology, London, England  
International Journal of Control, vol. 27, Feb. 1978, p. 261-269. Research supported by the Science Research Council of England.

**ABS:** The structure of linear, time-invariant, completely controllable and observable multivariable systems under the action of constant output feedback control laws is studied. A structure theorem is stated and proved. Following Dickinson (1976), the notion of covariant output feedback control laws is introduced. It is shown that these results can be used to investigate the problems concerning (1) the possibility of altering some of the Popov (1972) invariants by employing constant output feedback control, and (2) the asymptotic behavior of the closed loop poles under the variation of the output feedback gain matrix. 78/02/00 78A26249

**UTTL:** Geometric methods in the structural synthesis of linear multivariable controls  
**AUTH:** A/WOJHAM, W. M. PAA: A/Toronto, University, Toronto, Canada. In: Joint Automatic Control Conference, San Francisco, Calif., June 22-24, 1977. Proceedings, Volume 2, (A78-23851 08-53) New York, Institute of Electrical and Electronics Engineers, Inc., 1977, p. 594-600. National Research Council of Canada.

**ABS:** The last decade has brought significant growth in the application of geometric ideas to the formulation and solution of problems of controller synthesis. In this article the status of geometric state space theory is reviewed as developed for application to systems that are linear, multivariable and time-invariant. After a brief summary of the underlying geometric concepts ((A,B)-invariant subspaces and (A,B)-controllability subspaces) an outline is presented of two standard problems of feedback control that have been successfully attacked from this point of view. Certain

Issues of methodology are discussed, and some remarks on computational procedures are presented. 77/00/00 78A23876

**UTTL:** Using a complete block diagram for determining the characteristic equations of multivariable linear time-invariant control systems with lumped parameters  
**AUTH:** A/LATARNIK, M. PAA: A/Slask, Politechnika, Gliwice, Poland. Archiwum Automatyki i Telemechaniki, vol. 22, no. 4, 1977, p. 397-405. In Polish. 77/00/00 78A18031

**UTTL:** Multivariable linear time-varying tracking systems  
**AUTH:** A/DAMOWICZ, Z. B/PAWLOWSKI, K. PAA: B/Warszawa, Politechnika, Warsaw, Poland. Archiwum Automatyki i Telemechaniki, vol. 22, no. 4, 1977, p. 375-386. In Polish.

**ABS:** The paper presents a new method for the design of multivariable linear time-varying systems intended for tracking arbitrary signals generated by an autonomous linear system. To solve the tracking problem stated for such systems, properties of topologically equivalent systems and the theory of observers are used. 77/00/00 78A18029

**UTTL:** Transmission zeros of linear multivariable continuous-time systems  
**AUTH:** A/PORTER, B. B/DAZZO, J. J. PAA: A/Salford, University, Salford, England. B/USAF, Institute of Technology, Wright-Patterson AFB, Ohio. Electronics Letters, vol. 13, Nov. 24, 1977, p. 753-755.

**ABS:** It is shown possible for controllable and observable systems to characterize the set of transmission zeros in terms of the elements of the kernel of a singular pencil of matrices in a manner which yields directly the decomposition of the transfer function matrix and is especially effective in the determination of repeated transmission zeros which coincide with poles of the transfer function matrix. The characterization is equally applicable to continuous-time and discrete-time controllable and observable linear multivariable systems. As an illustration, the set of transmission zeros of the linearized representation of an orbiting satellite is investigated. 77/11/24 78A17342



**UTTL:** Computer-aided identification and multivariable control system design using convolution algebra

**AUTH:** A/GOUGH, N. E.; B/KLEFTHOORIS, D. N.; C/AL-THICA, R. S. PAA: C/Bradford, University, Bradford, England) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 357-364. 77/00/00 78A16324

**UTTL:** Controlled and conditioned invariant subspaces in analysis and synthesis of robust multivariable systems

**AUTH:** A/GHIBU, M. PAA: A/(Muenchen, Technische Universitaet, Munich, West Germany) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 317-321.

**ABS:** The known concept of controlled and conditioned invariant subspaces allows a stratification of the state, control (or input) and output space, which may be concisely represented in the form of two tables. An example of operations with some invariant subspaces is given for a robust system (output-insensitivity) whose eigenvalue remain unchanged. 77/00/00 78A16321

**UTTL:** The design of dynamic compensators for linear multivariable systems

**AUTH:** A/NAEIJUE, W. J.; B/BOSGRA, O. H. PAA: B/(Delft, Technische Hogeschool, Delft, Netherlands) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 205-213.

**ABS:** This paper studies optimal low order dynamic output feedback compensators for linear discrete-time constant systems. The general case is considered where some output variables are noise-free and some have measurement noise. The dynamic order of the compensator can be chosen arbitrarily in advance, and its parameters are determined in order that a quadratic performance index is minimized. The structure and performance of this class of compensators is analyzed. The relation with minimal-order observers is shown. The practical results that can be obtained are illustrated by an example. 77/00/00 78A16314

**UTTL:** Design of linear multivariable systems for stability under large parameter uncertainty

**AUTH:** A/SHAFED, U.; B/MACFARLANE, A. G. J. PAA: A/(Tel Aviv University, Tel Aviv, Israel); B/(Cambridge University, Cambridge, England) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 149-157.

**ABS:** The Sequential Return Difference method for the stabilization of linear multivariable systems is first extended for the case of perfectly known plant. Lower triangular controllers are introduced which prevent the need for constraints on the maximum allowed control effort. This approach is then extended to cope with large plant-parameter uncertainty where a systematic sequential-design technique is presented which guarantees the system stability for the whole uncertainty range. Under certain conditions, this technique also provides a means by which it is possible to convert a nonminimum-phase problem to a minimum-phase one, thus allowing the application of controllers with large gain that reduce the sensitivity of the system transference to parameter uncertainty. A synthesis technique is then introduced which is based on the sequential-design principle; this guarantees stability under uncertainty and meets pre-specified requirements on the sensitivity of the system performance at low frequencies. 77/00/00 78A16309

**UTTL:** Comparison sensitivity design of multivariable output feedback systems

**AUTH:** A/NAEIJUE, W. J.; B/BOSGRA, O. H. PAA: B/(Delft, Technische Hogeschool, Delft, Netherlands) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 127-135.

**ABS:** Research supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek. The concept of comparison sensitivity is considered for linear time-invariant multivariable systems and constant-gain or dynamic output feedback compensators. Necessary and sufficient conditions are derived for the existence of output feedback compensators which make the closed-loop system less sensitive to parameter variations than the nominally equivalent open-loop system, according to a particular integral-square sensitivity measure. The dynamic order of the compensator implicitly serves as a parameter in these conditions. A constructive procedure is shown to exist for the design of a sensitivity reducing

compensator and is illustrated by examples. 77/00/00  
78A16307

UTTL: A computer aided design method of multivariable systems using output feedback  
AUTH: A/BAVOUMI, M. M.; B/DUFFIELD, T. L. PAA: B/(Queen's University, Kingston, Ontario, Canada) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 119-125. National Research Council of Canada

ABS: A computer program for implementing the synthesis of output feedback decoupled multivariable systems is described. Attention is given to the problem of the joint stability of the closed loop decoupled system and the feedback subsystem. The calculations which must be carried out by the computer are described in a step-by-step manner, and specific details of the program are given. 77/00/00 78A16306

UTTL: Diagonal dominance using function minimization algorithms

AUTH: A/LEININGER, G. G. PAA: A/(Toledo, University, Toledo, Ohio) CORP: Toledo Univ., Ohio. In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 105-112.

ABS: A new approach to the design of multivariable control systems using the inverse Nyquist array method is proposed. The technique utilizes a conjugate direction function minimization algorithm to achieve dominance over a specified frequency range by minimizing the ratio of the moduli of the off-diagonal terms to the moduli of the diagonal term of the inverse open loop transfer function matrix. The technique is easily implemented in either a batch or interactive computer mode and will yield diagonalization when previously suggested methods fail. The proposed method has been successfully applied to design a control system for a sixteenth order state model of the F-100 turbofan engine with three inputs. 77/00/00 78A16304

UTTL: The application of multivariable control theory to spacecraft attitude control

AUTH: A/LIKINS, P. PAA: A/(Columbia University, New York, N.Y.) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 638 p (for

1977. p. 11-20.

ABS: An appraisal is made of past, present, and future applications of the general theory of multivariable control to the problem of spacecraft attitude control. Evolving uses of modern estimation theory in this field are described, and ad hoc approaches to multivariable control are noted, but formal applications of optimal estimation and control theory are seen to be limited for the most part to paper studies. The primary obstacle to successful application is seen to be the reliance of the theory on the fidelity of the model, which is peculiarly complex for modern spacecraft. Future projections indicate a need for a broader theory which envelops the modeling problem. In order to meet the challenges of configuration and attitude control of planned spacecraft with distributed flexibility, distributed sensors, and distributed actuators for control. 77/00/00 78A16303

UTTL: Recent results on decentralized control of large scale multivariable systems

AUTH: A/DAVISON, E. J. PAA: A/(Toronto, University, Toronto, Canada) In: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints. (A78-16301 04-63) Oxford and New York, Pergamon Press, 1977. p. 1-10. National Research Council of Canada

ABS: A general description of the structural properties of the decentralized control problem arising from the study of large scale systems is made. In this problem, constraints on the structure of the information flow between the manipulated inputs of the system and measured outputs of the system are imposed, and the goal is to find conditions of existence of decentralized controllers for the system such that: (1) the composite system is controllable and observable from a given input and output, (2) the composite system can be stabilized, (3) there exists a solution to the robust servomechanism problem, and (4) there exists a solution to the decentralized tuning regulator problem. Some numerical examples are included; in particular a power system consisting of nine synchronous machines. 77/00/00 78A16302

UTTL: Multivariable technological systems: International Symposium, 4th, Fredericton, New Brunswick, Canada, July 4-8, 1977. Preprints  
AUTH: A/ATHERTON, D. P. PAA: A/(New Brunswick, University, Fredericton, Canada) Symposium sponsored by the International Federation of Automatic Control Oxford and New York, Pergamon Press, 1977. 638 p (for

**ABS:** Individual items see A78-16302 to A78-16334)  
Papers are presented on the application of multivariable control theory to spacecraft attitude control, on the design of tracking systems for a class of multivariable linear systems with slow and fast models, on the design of PID controllers with application to a gas turbine, and on adaptive observer and identifier design for multi-input multi-output systems. Consideration is also given to recursive identification of the parameters of a multivariable system, a turbofan engine controller using multivariate feedback, the application of polynomial techniques to the multivariable control of jet engines, and Viking Orbiter attitude control analysis.  
77/00/00 78A16301

**UTTL:** Singular perturbation methods in the design of full-order observers for multivariable linear systems  
**AUTH:** A/PORTER, B. PAA: A/(Salford, University, Salford, Lancs., England) International Journal of Control, vol. 26, Oct. 1977, p. 589-594.  
**ABS:** Singular perturbation methods are used to establish the conditions under which full-order observers can be designed so as to reconstruct the state vectors of multivariable linear systems containing 'parasitic' elements, and the resulting design procedure is illustrated by designing a full-order observer for a third-order system. This procedure for the design of full-order observers is the dual of the procedure developed by Porter (1976) for the design of stabilizing state-feedback controllers for multivariable linear systems containing 'parasitic' elements. 77/10/00 78A11479

**UTTL:** Singular perturbation methods in the design of stabilizing state-feedback controllers for multivariable linear systems  
**AUTH:** A/PORTER, B. PAA: A/(Salford, University, Salford, Lancs., England) International Journal of Control, vol. 26, Oct. 1977, p. 583-587. 77/10/00 78A11478

**UTTL:** Invariant zeros of multivariable systems - A geometric analysis  
**AUTH:** A/OWENS, D. H. PAA: A/(Sheffield, University, Sheffield, England) International Journal of Control, vol. 26, Oct. 1977, p. 537-548.  
**ABS:** The invariant zeros of a linear multivariable system S(A, B, C) are defined geometrically. A canonical form is derived which illustrates the physical source of zeros in terms of state feedback and observability. Upper bounds on the number of zeros are derived and

related to the structure of the system transfer function matrix. 77/10/00 78A11477

**UTTL:** Method of design of multivariable self-adjusting control systems  
**AUTH:** A/ELISEEV, V. D. (Avtomatika i Telemekhanika, Apr. 1977, p. 66-74.) Automation and Remote Control, vol. 38, no. 4, Sept. 10, 1977, pt. 1, p. 506-513. Translation.  
**ABS:** (For abstract see issue 16, p. 2754. Accession no. A77-35932) 77/09/10 78A10261

**UTTL:** Multivariable system control /Revised edition/  
**AUTH:** A/FOSSARD, A.; B/GUEGUEN, C. PAA: A/(OMERA, Centre d'Etudes et de Recherches de Toulouse, Toulouse, France) Amsterdam, North-Holland Publishing Co., 1977. 424 p. Translation

**ABS:** State-space representations are examined, taking into account concepts and forms of state-space, questions of controllability and observability, controllability, in the output space, and problems of reproducibility. Transfer function matrices and state representations are considered along with differential-operator representations and state-space forms, structures and representations, and the use of canonical structural forms in connection with the teleology of state-space representations. Aspects of interactive and noninteractive control are also discussed. Attention is given to general problems of compensations and the connection with fundamental concepts, compensation by state feedback, observer theory, direct output feedback by augmentation of the original system, concepts and definitions of interaction, general problems posed by noninteractive control systems, noninteraction by state-space techniques, noninteraction by operational methods, and the realization of noninteractive control schemes.  
77/00/00 77A50299

**UTTL:** Frequency-domain dual-locus method for design of multivariable control systems  
**AUTH:** A/STEEL, G. K.; B/LISTER, P. F. PAA: A/(Aston, University, Birmingham, England); B/(Sussex, University, Brighton, England) Institution of Electrical Engineers, Proceedings, vol. 124, Sept. 1977, p. 809-813.

**ABS:** A design method is presented for linear multivariable systems with cascade controllers. The method is restricted to general systems with two input and two output variables and is applied in the frequency domain using dual-locus plots. Loci are formed that

relate directly to the closed loop characteristic polynomial of the system and conditions are derived from an unambiguous indication of stability. The loci are chosen so that the effects of controller elements can be readily assessed, and this leads to an iterative procedure for controller design. The method has been applied to a position control system for a 2-axis gyroscope and is shown to yield high performance although retaining simplicity in the controller functions. 77/09/00 77A49538

UTTL: Design of linear time-invariant multivariable systems tracking continuously differentiable signals  
AUTH: A/PUSZ, J. PAA: A/(Warszawa, Politechnika, Warsaw, Poland) Archiwum Automatyki i Telemekhaniki, vol. 22, no. 3, 1977, p. 211-218. In Polish. 77/00/00 77A48154

UTTL: Design of PID controllers for multivariable systems

AUTH: A/SERAJI, H.; B/TAROKH, M. PAA: B/(Arya Mehr University of Technology, Tehran, Iran) International Journal of Control, vol. 26, July 1977, p. 75-83. Research supported by the Arya Mehr University of Technology.

ABS: The paper develops a simple method for the design of PID controllers for linear multivariable systems where the controllers use only the available system outputs. The resulting closed-loop system has a specified set of poles, and in the steady state the outputs follow step commands and reject disturbances of any form with constant final values. For an m-input, l-output system of order n, the PID controller can place 3m poles of the (n+l)th-order closed-loop system at any desired locations. The controller matrices are assumed to have unity-rank and are obtained from linear equations. The method is illustrated by an example. 77/07/00 77A44446

UTTL: Feedback realizations in linear multivariable systems

AUTH: A/BERGSSON, G. PAA: A/(Allmänna Svenska Elektriska AB, Vasteras, Sweden) IEEE Transactions on Automatic Control, vol. AC-22, Aug. 1977, p. 576-585. National Research Council of Canada

ABS: A feedback system consists of two objects, a fixed parent system and a controller connected to it. The relationship between external and internal descriptions of such systems is described. It is shown that redundancy in a feedback system can be expressed as a partial reduction with respect to controllability

and observability. Applying this reduction yields a controller of lowest possible order. It is also shown that the requirement of internal stability can be expressed as a requirement on the external description of the feedback system only. This leads to a nice formalism for internally stable control synthesis using rational matrices. This is demonstrated in two important control problems, the model matching problem and the algebraic regulator problem, and by examples. 77/08/00 77A43768

UTTL: A design technique for linear multivariable feedback systems

AUTH: A/MACFARLANE, A. G. J.; B/KOUVARITAKIS, B. PAA: B/(Cambridge University, Cambridge, England) International Journal of Control, vol. 25, June 1977, p. 837-874.

ABS: A systematic approach is developed for the design of linear multivariable feedback control systems based on a manipulation of the set of frequency-conscious eigenvalues and eigenvectors of an open-loop transfer-function matrix. The key idea behind the approach used is that of an approximately-commutative controller. An algorithm for approximation of a frame of complex vectors by a frame of real vectors is developed and plays a basic role in the systematic design approach. An example, based on industrial plant data, is given showing how the design method is used. 77/06/00 77A41429

UTTL: A new method for reducing multivariable systems  
AUTH: A/HICKIN, J.; B/SINHA, N. K. PAA: B/(McMaster University, Hamilton, Ontario, Canada) In: Modeling and Simulation, Volume 7 - Proceedings of the Seventh Annual Pittsburgh Conference, Pittsburgh, Pa., April 26-27, 1976, Part 1. (A77-38176 17-66) Pittsburgh, Pa.: Instrument Society of America, 1976, p. 259-263.

ABS: A method for generating partial realizations of linear, time-invariant, multivariable systems is presented. These partial realizations are then used as reduced order models from which sub-optimal controls are derived for the original system. The results are compared with those of sub-optimal control by aggregation. 76/00/00 77A38190

UTTL: The multivariable servomechanism problem from the input-output viewpoint

AUTH: A/FRANCIS, B. A. PAA: A/(Cambridge University, Cambridge, England) IEEE Transactions on Automatic Control, vol. AC-22, June 1977, p. 322-328.

ABS: The control system considered consists of a plant or

actuator controlled by a compensator processing the error between a reference command input and the plant output. The plant and compensator are modeled in input-output form by linear convolution equations. Closed-loop stability and tracking are characterized, and the problem encountered when these two properties are insensitive to small errors in the plant and compensator models is considered. Finally, these results are compared with recent results on the corresponding state-space problem 77/06/00 77A37983

UTTL: A synthesis method for multivariable adaptive control systems

AUTH: A/ELISEEV, V. D. Avtomatika i Telemekhanika, Apr. 1977, p. 66-74. In Russian.

ABS: A method of synthesizing automatic adaptive control systems is proposed for multivariable plants with several controls. It is based on breaking down the synthesis problem into the synthesis of the principal circuit and the synthesis of the self-adjusting loops for the parameters of the principal circuit. The solution is obtained by using a mismatch signal of the plant to obtain a parametrically invariant principal circuit that is insensitive to changes in the plant parameters. 77/04/00 77A35932

UTTL: Multivariable controlled systems

AUTH: A/KANAREV, L. E. Differentsial'nye Uravneniya, vol. 13, Mar. 1977, p. 559, 560. In Russian.

ABS: The paper proposes an algorithm for computing the minimum and maximum vector sets for an arbitrary constant square matrix. This leads to the generation of natural canonical forms of the initial matrix and determines the matrix of transition to it. The approach is applied to the analysis of a multivariable controlled system for which conditions of 'tying' and 'untying' are established. 77/03/00 77A35613

UTTL: Minimal-order compensators for decoupling and arbitrary pole placement in linear multivariable systems

AUTH: A/SIRISENA, H. R.; B/CHOI, S. S. PAA: B/Canterbury, University, Christchurch, New Zealand) International Journal of Control, vol. 25, May 1977, p. 755-767.

ABS: Linear multivariable systems that can be decoupled by state-variable feedback are considered. The problem of arbitrary pole placement and/or decoupling by means of output feedback via a dynamic compensator is first obtained through algebraic decoupling theory. Then an

optimization problem in the free compensator parameters is derived by defining a performance index that measures both the amount of coupling in the closed-loop system and, when pole placement is also desired, the error in the closed-loop pole locations. The theory necessary for solving this optimization problem by means of standard gradient algorithms is developed. Minimal-order solutions are obtained because the compensator structure considered is quite general. Two illustrative numerical examples are presented. 77/05/00 77A34457

UTTL: The intersection of the root-loci of

multivariable systems with the imaginary axis  
AUTH: A/SHAKED, U. PAA: A/Tel Aviv University, Tel Aviv, Israel) International Journal of Control, vol. 25, Apr. 1977, n. 603-607.

ABS: A scalar polynomial expression in the scalar feedback gain, that is used to evaluate the root-loci of a linear multivariable system, is derived. It is shown that the set of values of the scalar gain for which the root-loci intersect the imaginary axis (excluding the origin) is a subset of these polynomial roots. A linear equation in the feedback gain, whose solution yields the gain values for which the root-loci pass through the origin, is found. Together with the polynomial's expression, this equation provides an easy way of calculating the values of the feedback gain for which one or more of the closed-loop poles moves from one half plane to another. 77/04/00 77A32427

UTTL: The linear multivariable regulator problem

AUTH: A/FRANCIS, B. A. PAA: A/Cambridge University, Cambridge, England) SIAM Journal of Control and Optimization, vol. 15, May 1977, p. 486-505.

ABS: Control of a linear multivariable system in the presence of plant disturbance and reference signals is considered. Necessary and sufficient conditions for the existence of a synthesis, that is, a compensator that provides closed loop stability and output regulation, are established. A basic property of a feedback synthesis is also proved 77/05/00 77A32174

UTTL: Synthesis of multivariable linear systems on an orthogonal basis

AUTH: A/PEROV, V. P. Akademiya Nauk SSSR, Doklady, vol. 232, Jan. 11, 1977, p. 300-303. In Russian.

ABS: The synthesis of statistically optimal multivariable linear systems with allowance for the requirements placed on their dynamic properties can be accomplished

76/00/00 77A29383

by representing the input process on an orthogonal basis and using a generalized performance index. In the present paper, a solution for the optimal operator which minimizes the general performance index is obtained in the form of a product of three matrices - the basis matrix, the filtering matrix, and the signal transformation matrix. 77/01/11 77A31976

UTTL: Minimum interaction-minimum variance controller for multivariable systems

AUTH: A/SINHA, A. K. IEEE Transactions on Automatic Control, vol. AC-22, Apr. 1977, p. 274, 275.

ABS: Rosenbrock's Inverse Nyquist array method for achieving diagonal dominance in multivariable systems and Astrom's self-tuning regulator design method for getting minimum variance controller for single input-single output systems have been combined to obtain controllers for multivariable systems which will give minimum interaction as well as minimum variance in the output. 77/04/00 77A30823

UTTL: Graphical stability analysis of non-linear

multivariable control systems

AUTH: A/SHANKAR, S.; B/ATHERTON, D. P. PAA: B/(New Brunswick, University, Fredericton, Canada) International Journal of Control, vol. 25, Mar. 1977, p. 375-388. Research supported by the University of Brunswick; National Research Council of Canada

ABS: Based on the Jury and Lee criterion, two new graphical methods to determine the stability bounds for nonlinear multivariable systems are developed. It is shown that Cook's multivariable circle theorem is a special case of the first method. The methods are best used on an interactive graphics terminal. Examples are given to illustrate the computational procedure. 77/03/00 77A29668

UTTL: Continuation methods for stability analysis of

multivariable feedback systems

AUTH: A/SAEKS, R.; B/CHAO, K. S.; C/HUANG, E. C. PAA: C/(Texas Tech University, Lubbock, Tex.) In: Midwest Symposium on Circuits and Systems, 19th, Milwaukee, Wis., August 16, 17, 1976, Proceedings, (A77-29377 12-63) North Hollywood, Calif., Western Periodicals Co., 1976, p. 346-348.

ABS: Techniques for the application of a Nyquist stability criterion to a linear time-invariant, multivariable feedback system with open loop transfer function are described. The techniques used (the eigenvector approach and the Jacobian method) are continuation methods for computing the system's eigenvalue loci.

UTTL: Key control assessment for linear multivariable systems

AUTH: A/MICHAEL, G. J.; B/SOGLIERO, G. S. PAA: B/(United Technologies Research Center, East Hartford, Conn.) In: Conference on Decision and Control and Symposium on Adaptive Processes, 15th, Clearwater, Fla., December 1-3, 1976, Proceedings, (A77-28801 12-63) New York, Institute of Electrical and Electronics Engineers, Inc., 1976, p. 1120-1125.

ABS: Analytical procedures for a priori assessment of key control variables in linear multi-input multi-output dynamic systems are developed and evaluated. These procedures result in explicit, quantitative measures of control influence which allow a control designer to assess the relative importance of all control variables prior to actual feedback control design. In this manner the designer can select for dynamic modulation those controls which have the strongest influence on system dynamic response. The developed methodology is applied to a detailed fifth-order F100 gas turbine engine model. Predicted key control variables are compared with actual key control variables as determined through physical insight into system dynamics and operating experience. 76/00/00 77A28852

UTTL: Specifications and criteria for multivariable control system design

AUTH: A/NEE, D. H. In: Productivity: Proceedings of the Joint Automatic Control Conference, West Lafayette, Ind., July 27-30, 1976, (A77-28626 12-63) New York, American Society of Mechanical Engineers, 1976, p. 627-633.

ABS: Suitable specifications for the design of a large class of linear multivariable control systems are set out, and existing methods examined. Two variations of the popular Inverse Nyquist Array method of Rosenbrock are proposed which both allow a less restricted class of system to be designed. The first result says in effect that the detailed high frequency performance of the inverse plant transfer matrix is of no interest, and in particular it does not need to be diagonally dominant, providing the gains are not too large. The second more important result allows the sharing of dominance between loops. This gives the sensible result that triangular plants with arbitrarily large off-diagonal elements, present no difficulty. The method tend to design diagonal controllers if possible, each largely non-interacting, thus ensuring plant integrity to both transducer and actuator

failures. Some results concerning pairing of variables to aid the design are given. 76/00/00 77A28649

UTTL: Disturbance localization in multivariable control problems

AUTH: A/SHAH, S. L.; B/SEBORG, D. E.; C/FISHER, D. G.  
PAA: C/Alberta, University, Edmonton, Canada) In:  
Productivity, Proceedings of the Joint Automatic  
Control Conference, West Lafayette, Ind., July 27-30,  
1976. (A77-28626 12-63) New York, American Society of  
Mechanical Engineers, 1976, p. 618-626. Research  
supported by the National Research Council of Canada.

ABS: A necessary and sufficient condition for disturbance localization is derived in terms of the structure of the closed-loop system eigenvectors. This condition provides considerable insight and can be used in conjunction with existing eigenvector/eigenvalue assignment techniques to synthesize regulatory controllers with disturbance localization characteristics. Simulation results for three numerical examples demonstrate that the proposed design procedure is straightforward and produces disturbance localization controllers which are superior to conventional controllers. 76/00/00 77A28648

UTTL: Application of multivariable optimal control techniques to a variable area turbine engine

AUTH: A/BEATTIE, E. C.; B/SPOCK, W. R. PAA: B/United Technologies Corp., Pratt and Whitney Aircraft Group, East Hartford, Conn.) In: Productivity, Proceedings of the Joint Automatic Control Conference, West Lafayette, Ind., July 27-30, 1976. (A77-28626 12-63) New York, American Society of Mechanical Engineers, 1976, p. 124-132.

ABS: The paper describes the application of a linear quadratic synthesis technique to the control design problem for an advanced turbofan engine with variable area turbines. The approach followed was to implement a combined matrix Riccati equation solution and a general linear dynamic simulation program as a fully interactive program on an IBM 370 computing system. Linear models generated from the nonlinear engine simulation were then input to this program and feedback gain matrices were calculated. Implementing the resulting control on the nonlinear engine required modifications to accommodate nonlinearities and control saturation, and to reduce control mode complexity. The control mode structure resulting from this design process is discussed and compared with a control mode developed using classical control design techniques. Representative transient results obtained

with both control modes are also presented. 76/00/00 77A28631

UTTL: Second-order eigenvalue sensitivities applied to multivariable control systems

AUTH: A/EL-DIN, H. M. Z.; B/ALDEN, R. T. H.; C/CHAKRAVARTI, P. C. PAA: C/McMaster University, Hamilton, Ontario, Canada) IEEE, Proceedings, vol. 65, Feb. 1977, p. 277, 278.

ABS: Second-order eigenvalue sensitivities are derived in terms of system parameters, and a method is developed for applying them to multivariable control systems described by state-space equations. Use of the method is demonstrated by considering the dynamic stability of a portion of a power system in which different models are employed for a composite load. In this example, the active power of the load is expressed as an exponential function of voltage with the exponent varied so that there is a nonlinear relationship between stability and a specific parameter. 77/02/00 77A24862

UTTL: Decoupling of multivariable systems using output feedback

AUTH: A/EL-BAGOURY, M. A.; B/BAYOUMI, M. M. PAA: B/Queen's University, Kingston, Ontario, Canada) IEEE Transactions on Automatic Control, vol. AC-22, Feb. 1977, p. 146-149. National Research Council of Canada

ABS: Decoupling of linear, time-invariant multivariable systems described by a rectangular transfer function matrix  $T(s)$ , using linear output feedback into single input-multiple output (SIMO) subsystems is analyzed. A set of necessary and sufficient conditions is derived through the structure of the desired transfer function matrix. 77/02/00 77A24198

UTTL: Computation of optimal output feedback controls for unstable linear multivariable systems

AUTH: A/CHOI, S. S.; B/SIRISENA, H. R. PAA: B/Canterbury University, Christchurch, New Zealand) IEEE Transactions on Automatic Control, vol. AC-22, Feb. 1977, p. 134-136.

ABS: Optimal output feedback control of linear multivariable systems for quadratic performance indices is considered. An iterative algorithm for computing the optimal feedback gains is proposed that does not require a stabilizing output feedback law for initialization, rather any stabilizing state feedback law suffices. An illustrative numerical example is included. 77/02/00 77A24196

- UTTL:** The generalized Nyquist stability criterion and multivariable root loci
- AUTH:** A/MACFARLANE, A. G. J.; B/POSTLETHWAITE, I. PAA: B/(Cambridge University, Cambridge, England) International Journal of Control, vol. 25, Jan. 1977, p. 81-127.
- ABS:** The background to the generalized Nyquist stability criterion for linear multivariable feedback systems is discussed. This leads to a proof based on the use of the principle of the Argument applied to an algebraic function defined on an appropriate Riemann surface. It is shown how the matrix-valued functions of a complex variable which define the loop transmittance, return-ratio, and return-difference matrices of feedback-systems analysis may be associated with a set of characteristic algebraic functions, each associated with a Riemann surface. These functions enable the characteristic loci to be put on a sound basis. The relationship between the algebraic structure of the matrix-valued functions and the appropriate complex-variable theory is carefully discussed. These extensions of the complex-variable concepts underlying the Nyquist criterion are then related to an appropriate generalization of the root locus concept. It is shown that multivariable root loci are the 180-deg phase loci of the characteristic functions of a return-ratio matrix on an appropriate Riemann surface, plus some possibly degenerate loci consisting of a single point each. 77/01/00 77A20924
- UTTL:** Design of multivariable linear time-invariant tracking systems in the presence of random disturbances
- AUTH:** A/PUSZ, J. PAA: A/(Warszawa, Politechnika, Warsaw, Poland) Archiwum Automatyki i Telemechaniki, vol. 21, no. 4, 1976, p. 415-423. In Polish.
- ABS:** The paper describes the state-space approach to the design of linear multivariable, time-invariant, continuous-time systems subjected to random disturbances, which track, in the sense of the first two moments, multivariable command signals which are functions of time. The class of disturbances consists of polynomial functions of time with random coefficients. An algorithm is given for determining the proportional element matrix such that the steady-state error is zero. 76/00/00 77A20156
- UTTL:** Decentralized control of linear multivariable systems
- AUTH:** A/CORFMAT, J. P.; B/MORSE, A. S. PAA: B/(Yale University, New Haven, Conn.) Automatica, vol. 12, Sept. 1976, p. 479-495. National Research Council of Canada
- ABS:** This paper studies the effect of decentralized feedback on the closed-loop properties of jointly controllable, jointly observable k-channel linear systems. Channel interactions within such systems are described by means of suitably defined directed graphs. The concept of a complete system is introduced. Complete systems prove to be precisely those systems which can be made both controllable and observable through a single channel by applying nondynamic decentralized feedback to all channels. Explicit conditions are derived for determining when the closed-loop spectrum of a k-channel linear system can be freely assigned or stabilized with decentralized control. 76/09/00 77A18275
- UTTL:** On the stability problem of multivariable model-following systems
- AUTH:** A/KUDVA, P.; B/GOURISHANKAR, V. PAA: B/(Alberta University, Edmonton, Canada) International Journal of Control, vol. 24, Dec. 1976, p. 801-805. National Research Council of Canada
- ABS:** The problem of stability of multivariable model-following systems is investigated. A condition for a plant with state feedback to follow a lower-order matchable model in a stable fashion is derived and illustrated by means of an example. 76/12/00 77A17084
- UTTL:** Linear multivariable control - Its development, computer implementation, and application to hydrofoil motion control problems
- AUTH:** A/DIXON, M. PAA: A/(Cambridge University, Cambridge, England) In: Hovering craft, hydrofoil and advanced transit systems: Proceedings of the Second International Conference, Amsterdam, Netherlands, May 17-20, 1976. (A77-17026 05-85) London, Kluwer Publications, 1976, p. 369-391.
- ABS:** The development and current state of the art of linear multivariable control theory is reviewed. A qualitative description is then given of an approach to the implementation of control system design techniques using an interactive dedicated minicomputer. The published design studies of hydrofoil craft motion controllers are critically examined and conclusions are drawn concerning other methods. The hydrofoil craft motion modelling and



simulation work at Cambridge is illustrated by graphical computer output. 76/00/00 77A17051

**UTTL:** A geometric approach to the inversion of multivariable systems

**AUTH:** A/KOUVARITAKIS, B. PAA: A/Cambridge University, Cambridge, England. International Journal of Control, vol. 24, Nov. 1976, p. 609-626.

**ABS:** The input and output matrix maps B and C of a linear multivariable system  $S(A, B, C)$  play an important role in determining the behaviour of the system. Square systems with the product  $CB$  full rank possess a simple state-space geometry which is deployed in the present paper for the derivation of an explicit state-space characterization of the inverse system. An efficient algorithm for the inversion of a system is then obtained. The elegance of the results discussed enables the examination of the duality between poles/modes and zeros-zero-directions. Finally, an extension of the above to systems with  $CB$  rank-deficient is undertaken. 76/11/00 77A11879

**UTTL:** Optimal linear control (characterization and loop transmission properties of multivariable systems

**AUTH:** A/HARVEY, C. A.; B/DOYLE, J. C. CORP: Honeywell Systems and Research Center, Minneapolis, Minn.

**RPT#:** AD-A075806 ONR-CR215-238-3 TR-3 HONEYWELL-78SRC73 78/08/01 80N74867

**UTTL:** An engineering approach to multivariable control system design

**AUTH:** A/HAMMAM, R. S. CORP: New South Wales Univ., Sydney (Australia). 75/04/00 80N70398

**UTTL:** Continuation methods for stability analysis of multivariable feedback systems

**AUTH:** A/SAEKS, R.; B/CHAO, K. S.; C/HUANG, E. C. CORP: Texas Technological Univ., Lubbock. CSS: (Dept. of Electrical Eng.) Presented at 19th Midwest Symp. on Circuits and Systems, 16-17 Aug. 1976

**RPT#:** AD-A036720 AFOSR-77-0070TR 76/08/00 77N82130

**UTTL:** Synthesis of observers for linear multivariable control systems

**AUTH:** A/GULLY, S. W. CORP: Polytechnic Inst. of New York. 76/00/00 77N70902

**UTTL:** Practical methods for the compensation and control of multivariable systems

**AUTH:** A/WOLOVICH, W. A.; B/FALB, P. L. CORP: Brown Univ., Providence, R. I. CSS: (Dept. of Engineering.)

**ABS:** A new parameter adaptive control scheme for linear multivariable systems has been developed. It is felt that such an adaptive controller could be used in a variety of aerospace applications. A "multi-purpose" controller has been designed which simultaneously decouples, places poles arbitrarily, rejects disturbances, insures zero error tracking, and is robust with respect to parameter variations. A new and straightforward method for obtaining simple low order models of systems whose dynamical behavior approximates that of more complex, higher order systems has also been developed. Such low order models can be used in the design of low compensators for the more complex systems. A complete new resolution has been presented to the question of what changes occur to the individual transfer matrix of elements of a linear multivariable system under local, scalar output feedback. In particular, it has been shown what poles become controllable and observable via any input/output pair when constant gain output feedback is applied between any (i-th) output and any (j-th) input. The question of parameter variation and the development of compensators which are insensitive to that variation has been resolved for a specific feedback group and is being studied for systems with parameters. Results have also been obtained for the pole-assignment problem involving parameters using intersection theory and some preliminary work has been done on the realization, coprime factorization and trace assignment problems for systems defined over the polynomial ring in n-variable over the integers.

**RPT#:** AD-A093126 AFOSR-80-12261R 80/10/31 81N17759

**UTTL:** Multivariable Nyquist array method with application to turbofan engine control

**AUTH:** A/LEININGER, G. G. CORP: Purdue Univ., Lafayette, Ind. CSS: (School of Mechanical Engineering.) In NASA, Lewis Research Center, Propulsion Controls, 1979 p 105-110 (SEE NB1-12090 03-07)

**ABS:** Extensions to the multivariable Nyquist array (MNA) method are used to design a feedback control system for the quiet clean shonhaul experimental engine. The results of this design are compared with those obtained from the deployment of an alternate control system design on a full scale nonlinear, real time digital simulation. The results clearly demonstrate the utility of the MNA synthesis procedures for highly nonlinear sophisticated design applications.

80/10/00 81N12101

UTTL: Recent developments in the Robustness Theory of Multivariable Systems. Proceedings of the ONR/MIT Workshop

AUTH: A/SANDELL, N. R., JR. CORP: Massachusetts Inst. of Tech., Cambridge, CSS: (Information and Decision Systems Lab.) Workshop held in Cambridge, Mass., 25-27 Apr. 1979

ABS: This report summarizes the recent developments in the Robustness Theory of Multivariable Systems. The motivation for the workshop was the dissemination and discussion of recent research results concerning the characterization of robust multivariable feedback systems, defined here as systems that perform satisfactorily despite differences between the mathematical model used for control synthesis and the actual dynamics, and the design of such systems. The report describes the motivation for the theoretical developments and the nature of the results that have been obtained. An introductory discussion of the papers presented at the workshop is provided; the papers themselves are included in an Appendix. Finally, the research directions identified in a roundtable discussion held at the end of the workshop are summarized.

RPT#: AD-A082362 LIDS-R-954 ONR-CR-215-271-1F 79/08/31  
80N25643

UTTL: Practical methods for the compensation and control of multivariable systems

AUTH: A/WOLOVICH, W. A.; B/FALB, P. L. CORP: Brown Univ., Providence, R. I. CSS: (Center for Dynamical Systems.)

ABS: A new method for obtaining simple, low order models for higher order stable systems was developed and tested on a sixteenth order model of the F-100 jet engine. A scalar adaptive control procedure was extended to the multivariable case. A polynomial matrix characterization of the maximal (A,B)-invariant and controllability subspace in the kernel of C was determined together with an algorithm for the state feedback controllers which yield such maximal subspaces. This work should have significant impact in the study of systems with imprecisely known parameters. A complete resolution to the problems of determining state feedback invariants and canonical forms for linear systems characterized by proper rational transfer matrices was obtained. A number of results have been obtained illustrating the richness of the linkage between system theory and algebraic-geometry. For example, it has been shown that any symmetric transfer matrix over reals has a symmetric realization (answering an old question in network theory). Finally, a new, general purpose

compensator for multivariable systems has been developed. This compensator insures simultaneous regulation, tracking, decoupling, stability, and robustness for a large class of linear multivariable systems.

RPT#: AD-A080265 AFOSR-80-0036TR 79/00/00 80N23053

UTTL: Robustness results in LQG based multivariable control designs

AUTH: A/LEHTOMAKI, N. A.; B/SANDELL, N. R., JR.; C/ATHANS, M. PAA: B/(Alpha Tech. Inc., Burlington, Mass.) CORP: Massachusetts Inst. of Tech., Cambridge, CSS: (Lab. for Information and Decision Systems.)

ABS: The robustness of control systems with respect to model uncertainty is considered using simple frequency domain criteria. Results are derived under a common framework in which the minimum singular value of the return difference transfer matrix is the key quantity. In particular, the LQ and LQG robustness results are discussed.

RPT#: LIDS-P-976 NASA-CR-162813 80/02/00 80N18769

UTTL: A summary of spectral synthesis procedures for multivariable systems

AUTH: A/LIBERTY, S. R.; B/WIELKE, R. R.; C/MAYNARD, R. A. CORP: Old Dominion Univ., Norfolk, Va.

ABS: A new approach to the eigensystem assignment problem is presented. The approach utilizes a null-space formulation of the eigenvalue/eigenvector assignment problem to simultaneously realize arbitrary eigenvalue specifications, approximate desired modal behavior, and achieve low eigensystem sensitivity with respect to plant parameter variations. The methods are applied to the design of regulator and integral plus proportional servo control systems.

RPT#: NASA-CR-162436 79/11/00 80N11862

UTTL: Frequency-domain approaches to linear multivariable control systems designs

AUTH: A/WEL, Y. J. P. CORP: Houston Univ., Tex. ABS: An effective method for multivariable control system designs is developed. Modeling control systems using industrial control specifications and model reduction using the mixed methods are described. Relationships between the classical control specifications and the optimal weighting factors and multivariable control system design using a noninteractive method are discussed. 79/00/00 80N10472

is also fully resolved. The procedures employed provide new insight regarding some related disturbance rejection questions. Numerous examples are presented in order to illustrate and clarify the presentation.

RPT#: AD-A050364 AFOSR-78-0168TR 77/11/00 78N20830

UTTL: Evaluation of an F100 multivariable control using a real-time engine simulation

AUTH: A/SZUCH, J. R.; B/SKIPRA, C.; C/SOEDER, J. F. PAA: B/IAFSC) CORP: National Aeronautics and Space Administration, Lewis Research Center, Cleveland, Ohio. Presented at 13th Propulsion Conf., Orlando, Fla., 11-13 Jul. 1977; sponsored by AIAA and Soc. of Automotive Engrs.

ABS: A multivariable control design for the F100 turbofan engine was evaluated, as part of the F100 multivariable control synthesis (NVCS) program. The evaluation utilized a real-time, hybrid computer simulation of the engine and a digital computer implementation of the control. Significant results of the evaluation are presented and recommendations concerning future engine testing of the control are made.

RPT#: NASA-TM-X-73648 E-9155 77/00/00 78N10097

UTTL: Diagonal dominance for the multivariable Nyquist array using function minimization

AUTH: A/LEININGER, G. G. CORP: Toledo Univ., Ohio. ABS: A new technique for the design of multivariable control systems using the multivariable Nyquist array method was developed. A conjugate direction function minimization algorithm is utilized to achieve a diagonal dominant condition over the extended frequency range of the control system. The minimization is performed on the ratio of the modulus of the off-diagonal terms to the modulus of the diagonal terms of either the inverse or direct open loop transfer function matrix. Several new feedback design concepts were also developed, including: (1) dominance control parameters for each control loop; (2) compensator normalization to evaluate open loop conditions for alternative design configurations; and (3) an interaction index to determine the degree and type of system interaction when all feedback loops are closed simultaneously. This new design capability was implemented on an IBM 360/75 in a batch mode but can be easily adapted to an interactive computer facility. The method was applied to the Pratt and Whitney F100 turbofan engine.

RPT#: NASA-CR-153058 TR-7701 77/05/00 77N24382

UTTL: An interactive classical control design algorithm for multivariable systems in state-space form

AUTH: A/LEE, G. K. F. CORP: Connecticut Univ., Storrs.

ABS: A simple design algorithm which blends the mathematical elegance of state-space theory with the graphical simplicity of root locus techniques is presented. An algorithm was developed for the constant output feedback problem which incorporates Jordan blocks to expose the influence of the gain matrix on the poles of the system. The methodology is extended to  $P$  control and general filter design. Plant and compensator dynamics are described in such a way that the influence of compensation on the poles of the system is directly observed. Zero placement is discussed and a general control design package is developed which implements the methodology. Examples are provided to illustrate the simplicity of the algorithm. 78/00/00 79N24670

UTTL: Multivariable control systems, finite-state linear sequential machines, and projective geometries. Some explicit interconnections

AUTH: A/ZALNAI, G. J. CORP: Georgia Inst. of Tech., Atlanta.

ABS: Deterministic time-invariant finite-state linear sequential machines are treated as discrete-time control systems over the finite field  $GF(q)$ . Adopting a modern multivariable control theory approach and a finite-geometric point of view, various structural aspects of linear machines are investigated. Numerous equivalent formulations, the relationship among state reachability, structural invariants, canonical forms, and state feedback are discussed. The concept of state observability is investigated and its relationship to state reachability is established through a duality theorem. 78/00/00 78N32771

UTTL: Practical methods for the compensation and control of multivariable systems

AUTH: A/FALB, P. L.; B/WOLOVICH, W. A. CORP: Brown Univ., Providence, R. I. CSS: (Lefschetz Center for Dynamical Systems.)

ABS: New techniques are developed for achieving a maximum number of zero entries in the closed loop transfer matrices of left invertible, linear multivariable systems through the employment of linear state variable feedback in conjunction with input dynamic compensation. Necessary and sufficient conditions are also presented for obtaining desired noninteractive structures for this class of systems. The question of simultaneous stabilization of the decoupled structures

UTTL: On the flexibility offered by state feedback in multivariable systems beyond closed-loop Eigenvalue assignment

AUTH: A/MOORE, B. C. CORP: Louisiana Office of State Planning, Baton Rouge. CSS: (Dept. of Computer Science.)

ABS: A characterization is given for the class of all closed loop eigenvector sets which can be obtained with a given set of distinct closed loop eigenvalues using state feedback. It is shown, furthermore, that the freedom one has in addition to specifying the closed loop eigenvalues is precisely this: to choose one set of closed loop eigenvectors from this class. Included in the proof of this result is an algorithm for computing the matrix of feedback gains which gives the chosen closed loop eigenvalues and eigenvectors. A design scheme based on these results is presented which gives the designer considerable freedom to choose the distribution of the modes among the output components. One interesting feature is that the distribution of a mode among the output components can be varied even if the mode is not controllable.

RPT#: AD-A033056 CONTROL-SYSTEMS-7609 AFOSR-76-11791R 76/04/00 77N22880

UTTL: Recent progress in multivariable control

AUTH: A/MONHAM, W. M. CORP: Toronto Univ. (Ontario). In NASA: Ames Res. Center Systems Reliability Issues for Future Aircraft p 157-164 (SEE N77-22808 13-59)

ABS: The major results in structural synthesis and design which are now available in textbook form are discussed which suggests a number of promising directions for future research. 75/00/00 77N22818

UTTL: Pre- and post-compensation to achieve system type in linear multivariable servomechanisms

AUTH: A/WOLFE, C. A.; B/MEDITCH, J. S. PAA: A/(Orincon. San Diego, Calif.) CORP: California Univ., Irvine. CSS: (School of Engineering.) Proc. of The 1976 Joint Autom. Control Conf., West Lafayette, Ind., 27-30 Jul. 1976; sponsored by Am. Inst. of Chem. Engr., ASME, IEEE Instrument Soc. of Am., Soc. Mfg. Engr.

ABS: Methods for specifying pre- and post-compensation to achieve a desired system type in linear multivariable servomechanisms are developed. Sufficient conditions are derived for both forms of compensation when the compensator is restricted to a diagonal input-output representation, and a method for realizing reduced-order precompensators is presented for the case where the restriction is removed.

RPT#: AD-A032873 AFOSR-76-11911R 76/07/00 77N22495

UTTL: Theory of system type for linear multivariable servomechanisms

AUTH: A/MEDITCH, J. S.; B/WOLFE, C. A. PAA: B/(Orincon. Inc., San Diego, Calif.) CORP: California Univ., Irvine. CSS: (School of Engineering.) Proc. of 1976 Joint Autom. Control Conf., West Lafayette, Ind., 27-30 Jul. 1976; sponsored by Am. Inst. of Chem. Engr., ASME, IEEE, Instrument Soc. of Am., and Soc. Mfg. Engr.

ABS: A theory of system type for linear multivariable servomechanisms is developed. New characterizations of system type are formulated and two systematic methods for determining system types are derived. Both methods are algorithmic and easier to apply than the techniques given in earlier studies. The first applies to the case where the open-loop system is described by its transfer function matrix, the second where its time domain state-space representation is given. New results are also presented for characterizing and identifying the system type for certain composite systems from a knowledge of the individual subsystem types.

RPT#: AD-A032874 AFOSR-76-11921R 76/07/00 77N22494

UTTL: A modified configuration for linear multivariable servomechanisms

AUTH: A/MEDITCH, J. S.; B/WOLFE, C. A. CORP: California Univ., Irvine. CSS: (School of Engineering.) Presented at the 13th Ann. Allerton Conf. on Circuit and System Theory, Montecello, Ill., 1-3 Oct. 1975; sponsored by Ill. Univ., Urbana-Champaign

ABS: Recent results on system type for linear multivariable servomechanisms are applied to a previously proposed type r multivariable servomechanism structure to obtain a new configuration. The latter possesses design freedom not present in the first configuration; this permits optimization of certain transient response properties of the closed-loop system without affecting previously specified system type and closed-loop pole location. An example is presented which demonstrates over a 35% improvement in integral-square tracking error relative to that for the original configuration.

RPT#: AD-A026297 AFOSR-76-01741R 75/10/00 77N14494

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15. Abstract			
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